

FREE AND BASED PATH GROUPOIDS

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ABSTRACT. We give an explicit description of the free path and loop groupoids in the Morita bicategory of translation topological groupoids. We prove that the free path groupoid of a discrete group acting on a topological space X is a translation groupoid given by the same group acting on the topological path space X^I . We give a detailed description of based path and loop groupoids and show that both are equivalent to topological spaces. We also establish the notion of homotopy and fibration in this context.

1. INTRODUCTION

Our aim is to give an explicit description of the path object in the bicategory of translation topological groupoids. Our main application will be in the setting of orbifolds as groupoids.

We adopt the model developed by Moerdijk and Pronk [9] to describe orbifolds in terms of groupoids. Essentially an orbifold is a Morita equivalence class of groupoids of a certain type, that we will call orbifold groupoids.

In this spirit, the right notion of morphism between orbifold groupoids is that of a generalized map. These generalized maps arise as morphisms in the bicategory of topological groupoids, functors and natural transformations when inverting the essential equivalences [12].

Most orbifolds - possibly all - can be represented by a groupoid given by a certain type of action of a group G on a topological space X . In this paper we will focus on these kind of groupoids $G \ltimes X$ representing orbifolds, called translation groupoids. In particular we will be interested in *developable* orbifolds defined by a discrete group acting properly on a space.

For these orbifolds, we use their groupoid characterization to obtain a description of the generalized maps from the interval to the orbifold as a translation groupoid. We prove that the free path groupoid of the translation groupoid $G \ltimes X$ is the translation groupoid $G \ltimes X^I$. In fact we describe three different approaches resulting in three characterizations of the path groupoid: as a colimit of G -paths, as a groupoid of multiple G -paths and as a translation groupoid $G \ltimes X^I$. We prove that the three groupoids are equivalent.

We show that this construction of the path groupoid is functorial and invariant under Morita equivalence.

The groupoid pullback of this model gives us the free loop groupoid which coincides with the descriptions given by Lupercio and Uribe [7], Adem, Leida and Ruan [1] and Noohi [10] in various contexts.

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Moreover, we use this model to calculate the based groupoid of paths between two points. We prove that this groupoid is actually equivalent to a topological space.

Using our description of the path groupoid, we provide an explicit characterization for a homotopy between two generalized maps, as well as a definition for orbifold fibrations. The path-loop morphism of a groupoid from its based path groupoid, is a groupoid fibration in this context.

We organize the paper in the following manner. In Section 2 we present some basic definitions and constructions for topological groupoids. We define translation groupoids and introduce the bicategory of translation groupoids resulting from inverting the essential equivalences. Section 3 is devoted to the construction of the free path groupoid. We give here an explicit equivalence between all models for the path groupoid. We provide also in this section a detailed description of the based path and loop groupoids. Section 4 concerns the characterization of the homotopy between generalized maps. In section 5 we provide a definition of groupoid fibration and prove that the relevant morphisms involving the various path groupoids are groupoid fibrations.

2. CONTEXT

2.1. Topological groupoids. A *topological groupoid* \mathcal{G} is a groupoid object in the category **Top** of topological spaces and continuous maps. Our notation for groupoids is that G_0 is the space of objects and G_1 is the space of arrows, with source and target maps $s, t : G_1 \rightarrow G_0$, multiplication $m : G_1 \times_{G_0} G_1 \rightarrow G_1$, inversion $i : G_1 \rightarrow G_1$, and object inclusion $u : G_0 \hookrightarrow G_1$.

The set of arrows from x to y is denoted $G(x, y) = \{g \in G_1 \mid s(g) = x \text{ and } t(g) = y\}$. The set of arrows from x to itself, $G(x, x)$, is a group called the *isotropy* group of \mathcal{G} at x and denoted G_x .

A *strict morphism* $\phi : \mathcal{K} \rightarrow \mathcal{G}$ of groupoids is a functor given by two continuous maps $\phi : K_1 \rightarrow G_1$ and $\phi : K_0 \rightarrow G_0$ that together commute with all the structure maps of the groupoids \mathcal{K} and \mathcal{G} .

A *natural transformation* $T : \phi \Rightarrow \psi$ between two morphisms $\phi, \psi : \mathcal{K} \rightarrow \mathcal{G}$ is a continuous map $T : K_0 \rightarrow G_1$ with $T(x) : \phi(x) \rightarrow \psi(x)$ such that for any arrow $h : x \rightarrow y$ in K_1 , the identity $\psi(h)T(x) = T(y)\phi(h)$ holds. Since we are in a topological groupoid and inversion is continuous, we also have a natural transformation $T^{-1} : \psi \Rightarrow \phi$ and write $\phi \sim_T \psi$.

Topological groupoids, strict morphisms and natural transformations form a 2-category that we denote **TopG**.

A strict morphism $\epsilon : \mathcal{K} \rightarrow \mathcal{G}$ of topological groupoids is an *essential equivalence* if

- (i) ϵ is essentially surjective in the sense that

$$s\pi_1 : G_1 \times_{G_0}^t K_0 \rightarrow G_0$$

is an open surjection where $G_1 \times_{G_0}^t K_0$ is the pullback along the target $t : G_1 \rightarrow G_0$;

- (ii) ϵ is fully faithful in the sense that K_1 is the following pullback of topological spaces:

$$\begin{array}{ccc} K_1 & \xrightarrow{\epsilon} & G_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ K_0 \times K_0 & \xrightarrow{\epsilon \times \epsilon} & G_0 \times G_0 \end{array}$$

Note that if there exists a functor $\delta : \mathcal{G} \rightarrow \mathcal{K}$ with natural transformations $\eta : \text{id}_{\mathcal{G}} \Rightarrow \epsilon \circ \delta$ and $\nu : \delta \circ \epsilon \Rightarrow \text{id}_{\mathcal{K}}$ in \mathbf{TopG} the functor ϵ is essentially surjective, indeed, $s\pi_1$ has a section defined by $(\eta_x, \delta(x)) : G_0 \rightarrow G_1 \times_{G_0}^t K_0$ which implies that it is open and surjective. ϵ is fully faithful because the map $K_1 \rightarrow K_0 \times_{G_0 \times G_0}^s \mathcal{G}_1$ has an inverse defined by $(x, y, h) \rightarrow \nu_y \circ \delta(h) \circ \nu_x^{-1}$.

An essential equivalence $\epsilon : \mathcal{K} \rightarrow \mathcal{G}$ does not generally have an inverse functor $\delta : \mathcal{G} \rightarrow \mathcal{K}$ such that $\epsilon \circ \delta \sim_T \text{id}_{\mathcal{G}}$ and $\delta \circ \epsilon \sim_{T'} \text{id}_{\mathcal{K}}$ in \mathbf{TopG} . The functor δ exists by the axiom of choice but in general it is not *continuous*.

Definition 2.1. Let $\psi : \mathcal{K} \rightarrow \mathcal{G}$ and $\phi : \mathcal{L} \rightarrow \mathcal{G}$ be strict morphisms. The *groupoid pullback* $\mathcal{P} = \mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is the topological groupoid whose space of objects is $P_0 = K_0 \times_{G_0}^t G_1 \times_{G_0}^s L_0$ and space of arrows is $P_1 = K_1 \times_{G_0}^t G_1 \times_{G_0}^s L_1$. Source and target maps are given by $s(k, g, l) = (s(k), \psi(k)^{-1}g\phi(l), s(l))$ and $t(k, g, l) = (t(k), g, t(l))$. There is a square of morphisms

$$\begin{array}{ccc} \mathcal{K} \times_{\mathcal{G}} \mathcal{L} & \xrightarrow{\pi_1} & \mathcal{K} \\ \pi_2 \downarrow & & \downarrow \psi \\ \mathcal{L} & \xrightarrow{\phi} & \mathcal{G} \end{array}$$

which commutes up to a natural transformation, and is universal with this property.

Definition 2.2. The groupoids \mathcal{K} and \mathcal{G} are *Morita equivalent* if there exists a groupoid \mathcal{L} and a span

$$\mathcal{K} \xleftarrow{\sigma} \mathcal{L} \xrightarrow{\epsilon} \mathcal{G}$$

where ϵ and σ are essential equivalences. We write $\mathcal{G} \sim_M \mathcal{K}$.

The proof that a Morita equivalence is an equivalence relation is based in the groupoid pullback defined above.

A *generalized map* from \mathcal{K} to \mathcal{G} is a span $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ such that ϵ is an essential equivalence. Two generalized maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ are *equivalent* if there exists a diagram

$$\begin{array}{ccccc} & & \mathcal{J} & & \\ & \epsilon \swarrow & \uparrow u & \searrow \phi & \\ \mathcal{K} & & \mathcal{L} & & \mathcal{G} \\ & \nwarrow \epsilon' & \downarrow v & \nearrow \phi' & \\ & & \mathcal{J}' & & \end{array}$$

\sim_T $\sim_{T'}$

which is commutative up to natural transformations and where \mathcal{L} is a topological groupoid, and u and v are essential equivalences.

2.2. The Morita bicategory of topological groupoids \mathbf{MTopG} . Consider the class of arrows E given by the essential equivalences in the 2-category \mathbf{TopG} . It was proven by Pronk in [12] that E satisfies the conditions to admit a bicalculus of fractions. The bicategory of fractions $\mathbf{TopG}(E^{-1})$ obtained by formally inverting the essential equivalences is what we call the *Morita bicategory of topological groupoids* and we denote \mathbf{MTopG} .

The explicit description of the bicategory \mathbf{MTopG} is as follows:

- Objects are topological groupoids \mathcal{G} .
- A 1-morphism from \mathcal{K} to \mathcal{G} is a *generalized map*

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

such that ϵ is an essential equivalence.

- A 2-morphism from $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ to $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ is given by a class of diagrams:

$$\begin{array}{ccccc} & & \mathcal{J} & & \\ & \epsilon \swarrow & \uparrow u & \searrow \phi & \\ \mathcal{K} & & \mathcal{L} & & \mathcal{G} \\ & \nwarrow \epsilon' & \downarrow v & \nearrow \phi' & \\ & & \mathcal{J}' & & \end{array}$$

\sim_T $\sim_{T'}$

where \mathcal{L} is a topological groupoid, and u and v are essential equivalences.

The horizontal composition of generalized maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{G} \xleftarrow{\zeta} \mathcal{J}' \xrightarrow{\psi} \mathcal{L}$ is given by the diagram

$$\begin{array}{ccccc} & & \mathcal{J}' \times_{\mathcal{G}} \mathcal{J} & & \\ & \swarrow & & \searrow & \\ & \mathcal{J} & & \mathcal{J}' & \\ \epsilon \swarrow & & \phi \searrow & \zeta \swarrow & \psi \searrow \\ \mathcal{K} & & \mathcal{G} & & \mathcal{L} \end{array}$$

where $\mathcal{J}' \times_{\mathcal{G}} \mathcal{J}$ is the weak pullback of groupoids. Note that this composition is associative only up to a 2-morphism.

2.3. Translation groupoids. Let G be a topological group with a continuous left action on a topological space X . Then the *translation groupoid* $G \ltimes X$ is defined by:

- The space of objects is X itself, and the space of arrows is the cartesian product $G \times X$.
- The source $s : G \times X \rightarrow X$ is the second projection, and the target $t : G \times X \rightarrow X$ is given by the action. Then (g, x) is an arrow $x \rightarrow gx$.
- The other structure maps are defined by the unit $u(x) = (e, x)$, where e is the identity element in G , and $(h, gx) \circ (g, x) = (h \star g, x)$ where \star is the group multiplication.

Example 2.3. These examples will appear later on in our applications.

- (1) *Unit groupoid.* Consider the groupoid $e \ltimes X$ given by the action of the trivial group e on the topological space X . This is a topological groupoid whose arrows are all units. In this way, any topological space can be considered as a groupoid.
- (2) *Multiplication groupoid.* Let H be a subgroup of a topological group G . Then H acts by multiplication on G .
- (3) *Conjugation groupoid.* Let H be a subgroup of a topological group G . Then H acts by conjugation on G .
- (4) *Point groupoid.* Let G be a topological group. Let \bullet be a point. Consider the groupoid $G \ltimes \bullet$ where G acts trivially on the point. This is a topological groupoid with exactly one object \bullet and G is the space of arrows in which the maps s and t coincide. We call $G \ltimes \bullet$ the point groupoid associated to G . In this way any group can be considered as a groupoid.

We will denote $\mathbf{1}$ the *trivial groupoid* with one object and one arrow, that is $\mathbf{1} = e \ltimes \bullet$ the unit groupoid over a point or a point groupoid associated to the trivial group.

2.4. The Morita bicategory of translation groupoids MTrG . We construct now a sub bicategory MTrG of the Morita bicategory of topological groupoids MTopG where the objects are strictly the translation groupoids and the maps are equivariant ones.

An *equivariant map* $G \ltimes X \rightarrow K \ltimes Y$ between translation groupoids consists of a pair $\varphi \ltimes f$, where $\varphi : G \rightarrow K$ is a group homomorphism and $f : X \rightarrow Y$ satisfies $f(gx) = \varphi(g)f(x)$ for $g \in G$ and $x \in X$.

Translation groupoids, equivariant maps and natural transformations form a 2-category that we denote TrG .

Proposition 2.4. [13] *Let $\psi : G \ltimes X \rightarrow L \ltimes Z$ and $\phi : H \ltimes Y \rightarrow L \ltimes Z$ be equivariant maps. The fibre product \mathcal{K}*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\pi_1 \ltimes f} & G \ltimes X \\ \pi_2 \ltimes g \downarrow & & \downarrow \psi \\ H \ltimes Y & \xrightarrow{\phi} & L \ltimes Z \end{array}$$

is again a translation groupoid. Moreover, its structure group is $G \times H$, $\mathcal{K} = (G \times H) \ltimes P$ and the first components of the equivariant maps $\pi_1 \ltimes f$ and $\pi_2 \ltimes g$ are the group projections $\pi_1 : G \times H \rightarrow G$ and $\pi_2 : G \times H \rightarrow H$.

An *equivariant essential equivalence* is an equivariant map $\xi \ltimes \epsilon$ which is an essential equivalence.

Consider the bicategory whose

- Objects are translation groupoids $G \ltimes X$.
- 1-morphisms from $G \ltimes X$ to $K \ltimes Y$ are *equivariant generalized maps*

$$G \ltimes X \xleftarrow{\xi \ltimes \epsilon} L \ltimes Z \xrightarrow{\varphi \ltimes f} K \ltimes Y$$

such that $\xi \ltimes \epsilon$ is an equivariant essential equivalence.

- A 2-morphism \Rightarrow from the equivariant generalized map $G \ltimes X \xleftarrow{\xi \ltimes \epsilon} L \ltimes Z \xrightarrow{\varphi \ltimes f} K \ltimes Y$ to $G \ltimes X \xleftarrow{\xi' \ltimes \epsilon'} L' \ltimes Z' \xrightarrow{\varphi' \ltimes f'} K \ltimes Y$ is given by a class of diagrams:

$$\begin{array}{ccccc}
 & & L \ltimes Z & & \\
 & \swarrow \xi \ltimes \epsilon & \uparrow u & \searrow \varphi \ltimes f & \\
 K \ltimes Y & \sim_T & R \ltimes U & \sim_{T'} & G \ltimes X \\
 & \swarrow \xi' \ltimes \epsilon' & \downarrow v & \searrow \varphi' \ltimes f' & \\
 & & L' \ltimes Z' & &
 \end{array}$$

where $R \ltimes U$ is a translation groupoid, and u and v are equivariant essential equivalences.

Translation groupoids, equivariant generalized map and diagrams as above form the *Morita bicategory of translation groupoids* that we denote \mathbf{MTrG} .

3. PATH GROUPOID

From now on, all groups considered will be *discrete* groups acting *properly* on a topological space. Orbifolds defined by a discrete group acting properly on a space are called *developable*.

3.1. G -paths. We define a path in the groupoid $G \ltimes X$ (in the Morita bicategory of topological groupoids) as a generalized map from the unit groupoid $I = [0, 1]$ to $G \ltimes X$. That is, a map (ϵ, α)

$$I \xleftarrow{\epsilon} I' \xrightarrow{\alpha} G \ltimes X$$

where I' is a groupoid essentially equivalent to the unit groupoid I .

By considering groupoid atlases, we can see that each class of equivalence of generalized maps $[I \xleftarrow{\epsilon} I' \xrightarrow{\alpha} G \ltimes X]$ has a representant of the form:

$$I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X$$

where I_{S_n} is the groupoid associated to a subdivision

$$S_n = \{0 = r_0 \leq r_1 < \cdots < r_{n-1} \leq r_n = 1\}$$

of the interval $I = [0, 1]$ as explained below.

The space of objects of the groupoid I_{S_n} is the disjoint union

$$\bigsqcup_{i=1}^n [r_{i-1}, r_i]$$

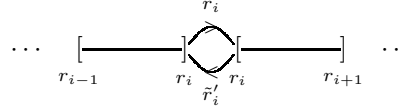
where we denote (r, i) an element r in the connected component $[r_{i-1}, r_i]$.

The space of arrows of I_{S_n} is given by the disjoint union

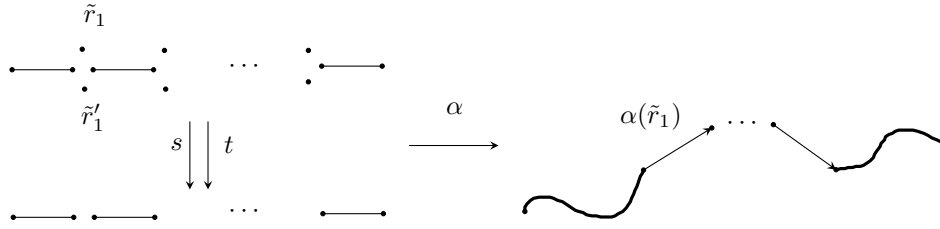
$$\left(\bigsqcup_{i=1}^n [r_{i-1}, r_i] \right) \bigsqcup \{ \tilde{r}_1, \dots, \tilde{r}_{n-1}, \tilde{r}'_1, \dots, \tilde{r}'_{n-1} \}$$

where $\bigsqcup_{i=1}^n [r_{i-1}, r_i]$ is the set of unit arrows and for each point r_i in the subdivision S_n two arrows were added: \tilde{r}_i and its inverse arrow \tilde{r}'_i such that the source of \tilde{r}_i is

(r_i, i) and its target is $(r_i, i + 1)$.



Definition 3.1. A G -path in the groupoid $G \ltimes X$ is a generalized map $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X$.



We will give an explicit characterization of the left leg $\epsilon : I_{S_n} \rightarrow I$ in the generalized map representing a G -path. On objects ϵ is the inclusion, $\epsilon(r, i) = r$ and on arrows it sends all arrows to identity arrows, $\epsilon(\tilde{r}_i) = \text{id}_{r_i}$.

Now we will describe the right leg $\alpha : I_{S_n} \rightarrow G \ltimes X$. On objects, we have a map $\bigsqcup_{i=1}^n [r_{i-1}, r_i] \rightarrow X$, hence a map in each connected component:

$$\alpha_i : [r_{i-1}, r_i] \rightarrow X$$

and on arrows we need also to map the points $\{\tilde{r}_1, \dots, \tilde{r}_{n-1}, \tilde{r}'_1, \dots, \tilde{r}'_{n-1}\}$ into $G \ltimes X$, let

$$\alpha(\tilde{r}_i) = (k_i, \alpha_i(r_i))$$

satisfying the condition $k_i \alpha_i(r_i) = \alpha_{i+1}(r_i)$.

Therefore we can give an explicit description of the set of maps from I_{S_n} to $G \ltimes X$ in terms of (reparametrized) paths on X and elements of the group G :

$$\text{Map}(I_{S_n}, G \ltimes X) = \{(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \in (X^I)^n \times G^{n-1} \mid k_i \alpha_i(r_i) = \alpha_{i+1}(r_i)\}.$$

Observe that this space can be obtained as the pullback of the following maps:

$$\begin{array}{ccc} & (X^I)^n & \\ & \downarrow (\text{ev}, \dots, \text{ev}) & \\ (G \times X)^{n-1} & \xrightarrow{(s, t)} & (X \times X)^{n-1} \end{array}$$

Then $\text{Map}(I_{S_n}, G \ltimes X) = (X^I)^n \times_{(X \times X)^{n-1}} (G \times X)^{n-1} = (X^I)^n \times_{X^{n-1}} G^{n-1}$ where reparameterizations are allowed.

We will establish now an equivalence relation between the generalized maps defining our G -paths which will allow us to give a groupoid structure to the set of G -paths.

Definition 3.2. Two G -paths $I \xleftarrow{\epsilon} I_{S_m} \xrightarrow{\alpha} G \ltimes X$ and $I \xleftarrow{\epsilon'} I_{S_{m'}} \xrightarrow{\beta} G \ltimes X$ are equivalent if there exist a subdivision S_n and essential equivalences u and v such that the following diagram commutes up to natural transformations.

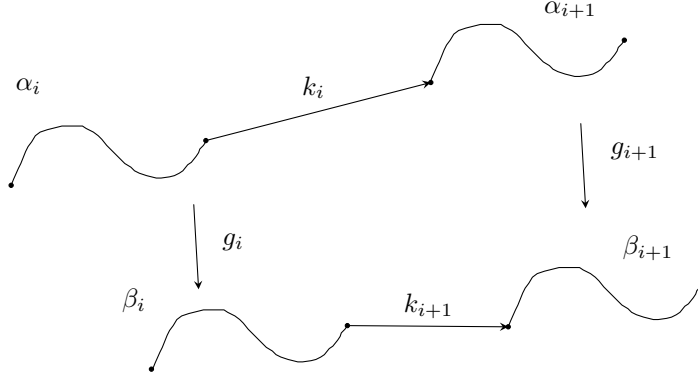
$$\begin{array}{ccccc}
 & & I_{S_m} & & \\
 & \epsilon \swarrow & \uparrow u & \searrow \alpha & \\
 I & \sim & I_{S_n} & \sim & G \ltimes X \\
 & \swarrow \epsilon' & \downarrow v & \searrow \beta & \\
 & & I_{S_{m'}} & &
 \end{array}$$

Since G is discrete, the condition $\alpha u \sim \beta v$ guarantee the existence of a natural transformation $T : \bigsqcup_{i=1}^n [r_{i-1}, r_i] \rightarrow G \times X$ such that $T(r, i) = (g_i, \alpha_i(r))$ with $\beta_i(r) = g_i \alpha_i(r)$. By naturality of the transformation we have that the following diagram commutes

$$\begin{array}{ccc}
 \alpha_i(r_i, i) & \xrightarrow{g_i} & \beta_i(r_i, i) \\
 k_i \downarrow & & \downarrow k'_i \\
 \alpha_{i+1}(r_i, i) & \xrightarrow{g_{i+1}} & \beta_{i+1}(r_i, i)
 \end{array}$$

therefore $k'_i = g_{i+1} k_i g_i^{-1}$ for all $i = 1, \dots, n-1$.

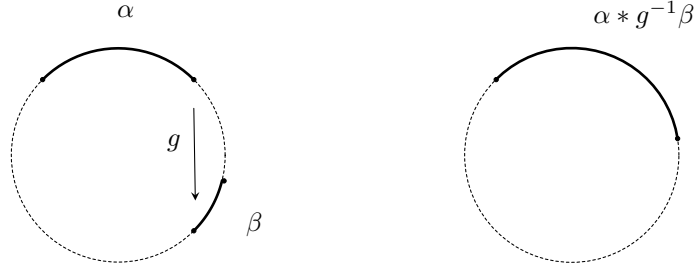
Remark 3.3. Two G -paths are equivalent if there exists a common subdivision S_n and $g_i \in G$ such that $\beta_i(r) = g_i \alpha_i(r)$ for all $i = 1, \dots, n$ and $k'_i = g_{i+1} k_i g_i^{-1}$ for all $i = 1, \dots, n-1$.



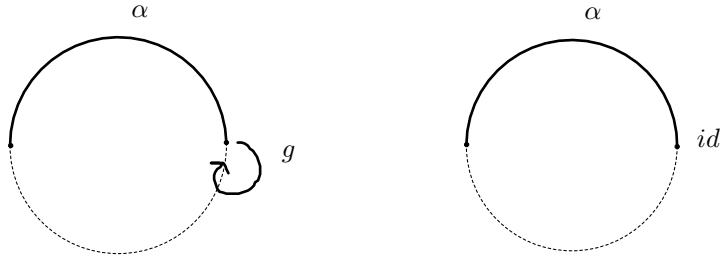
Then we have an action of G^n on the space $\text{Map}(I_{S_n}, G \ltimes X)$ which determines the translation groupoid $G^n \ltimes ((X^I)^n \times_{X^{n-1}} G^{n-1})$. Source and target are given by $s((g_1, \dots, g_n), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})) = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ and $t((g_1, \dots, g_n), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})) = (g_1 \alpha_1, \dots, g_n \alpha_n, g_2 k_1 g_1^{-1}, \dots, g_n k_{n-1} g_{n-1}^{-1})$.

Example 3.4. Let $G = \mathbb{Z}_2 = \{e, g\}$ act on S^1 by reflection. The following paths are equivalent:

- (1) We have that $(\alpha, g, \beta) \sim \alpha * g \beta$ since $(\alpha, g, \beta) \sim (\text{id } \alpha, g g \text{id}, g \beta) = (\alpha, \text{id}, g \beta)$.



- (2) Notice that in general $(\alpha, g) \sim \alpha$ for all $g \in G$.



These are relevant examples to have in mind in order to understand the difference between the free paths we just introduced and the based ones that will be introduced later as well as the difference between free paths and free loops.

In order to account for all possible subdivisions, we will consider the colimit of the groupoids $G^n \ltimes \text{Map}(I_{S_n}, G \ltimes X)$ over a partially ordered set that we describe next.

We define the category \mathcal{C}_I as the category with objects the sets $S_n = \{0 = r_0 \leq r_1 \leq \dots \leq r_n = 1\}$ and morphisms generated (as a category) by the set of morphisms,

$$d_i : \{0 = r_0 \leq r_1 \leq \dots \leq r_i \leq \dots \leq r_n = 1\} \rightarrow \{0 = r_0 \leq \dots \leq \widehat{r_i} \leq \dots \leq r_n = 1\}$$

where d_i drops the i -th element, subject to the relations $d_i d_j = d_{j-1} d_i$ for $1 \leq i < j \leq n-1$ and $d_i = d_j$ if $r_i = r_j$. Therefore we have a morphism from $\{r_0 \leq r_1 \leq \dots \leq r_n\}$ to $\{t_0 \leq t_1 \leq \dots \leq t_m\}$ if $\{t_0, t_1, \dots, t_m\} \subseteq \{r_0, r_1, \dots, r_n\}$ and the multiplicity of repeated elements increases, i.e. for every i , $|\{j \mid t_j = r_i\}| \leq |\{j \mid r_j = r_i\}|$. We call \mathcal{C}_I the category of subdivisions of I which is a cofiltered category.

The categories I_{S_n} are naturally indexed by the category \mathcal{C}_I , to the morphism $d_i : S_n \rightarrow S_{n-1}$ we assign the morphism $d_{i*} : I_{S_n} \rightarrow I_{S_{n-1}}$ that on objects concatenates $[r_{i-1}, r_i] \cup [r_i, r_{i+1}]$ to $[r_{i-1}, r_{i+1}]$ and on morphisms sends \tilde{r}_i and its inverse arrow \tilde{r}'_i to the identity arrow on r_i .

$$\begin{array}{ccc}
\begin{array}{c} \tilde{r}_i \\ \text{---} \circ \text{---} \\ \tilde{r}_i' \\ \text{---} \end{array} & \xrightarrow{d_{i*}} & \begin{array}{c} \text{---} | \text{---} \\ r_{i-1} \quad r_i \quad r_{i+1} \end{array}
\end{array}$$

Therefore we have a contravariant functor ψ from \mathcal{C}_I^{op} to topological spaces that on objects sends S_n to $\text{Map}(I_{S_n} \rightarrow G \ltimes X)$ and on morphisms sends $d_i : S_n \rightarrow S_{n-1}$ to the morphism $d_i^* : \text{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) \rightarrow \text{Map}(I_{S_n} \rightarrow G \ltimes X)$ given by taking $\alpha \in \text{Map}(I_{S_{n-1}} \rightarrow G \ltimes X)$ represented by $(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2})$ and sending it to $(\alpha_1, \dots, \alpha_i|_{[r_{i-1}, r_i]}, \alpha_i|_{[r_i, r_{i+1}]}, \dots, \alpha_{n-1}, k_1, \dots, k_{i-1}, id, k_i, k_{i+1}, \dots, k_{n-2})$, i.e. taking $\alpha_i : [r_{i-1}, r_{i+1}] \rightarrow X$ to the restrictions to $[r_{i-1}, r_i]$ and $[r_i, r_{i+1}]$. Then ψ is the composition of the following functors:

$$\mathcal{C}_I^{op} \longrightarrow \text{Gpd}^{op} \longrightarrow \text{Top}$$

$$\begin{array}{ccccc}
S_n & \longrightarrow & I_{S_n} & \longrightarrow & \text{Map}(I_{S_n} \rightarrow G \ltimes X) \\
\downarrow d_i & & \downarrow d_{i*} & & \uparrow d_i^* \\
S_{n-1} & \longrightarrow & I_{S_{n-1}} & \longrightarrow & \text{Map}(I_{S_{n-1}} \rightarrow G \ltimes X)
\end{array}$$

We have an action of G^n on $\text{Map}(I_{S_n} \rightarrow G \ltimes X)$ given by

$$(g_1, \dots, g_n) \cdot (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) = (g_1 \alpha_1, \dots, g_n \alpha_n, g_2 k_1 g_1^{-1}, \dots, g_n k_{n-1} g_{n-1}^{-1}).$$

The map $d_i^* : \text{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) \rightarrow \text{Map}(I_{S_n} \rightarrow G \ltimes X)$ is equivariant with respect to the map $\sigma_i : G^{n-1} \rightarrow G^n$ given by $\sigma_i(g_1, \dots, g_{n-1}) = (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$. This means that

$$\begin{aligned}
\sigma_i(g_1, \dots, g_{n-1}) \cdot d_i^*(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2}) \\
= d_i^*((g_1, \dots, g_{n-1}) \cdot (\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2}))
\end{aligned}$$

This is because $(g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$ acting on

$$(\alpha_1, \dots, \alpha_i|_{[r_{i-1}, r_i]}, \alpha_i|_{[r_i, r_{i+1}]}, \dots, \alpha_{n-1}, k_1, \dots, k_{i-1}, id, k_i, k_{i+1}, \dots, k_{n-2})$$

is equal in the first part to

$$(g_1 \alpha_1, \dots, g_i \alpha_i|_{[r_{i-1}, r_i]}, g_i \alpha_i|_{[r_i, r_{i+1}]}, \dots, g_{n-1} \alpha_{n-1})$$

and in the second part to

$$(g_2 k_1 g_1^{-1}, \dots, g_i k_{i-1} g_{i-1}^{-1}, g_i id g_i^{-1}, g_{i+1} k_i g_i^{-1}, \dots, g_{n-1} k_{n-2} g_{n-2}^{-1})$$

which is

$$(g_2 k_1 g_1^{-1}, \dots, g_i k_{i-1} g_{i-1}^{-1}, id, g_{i+1} k_i g_i^{-1}, \dots, g_{n-1} k_{n-2} g_{n-2}^{-1})$$

this is precisely

$$d_i^*((g_1, \dots, g_{n-1}) \cdot (\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2})).$$

Therefore we have a contravariant functor from \mathcal{C}_I to translation groupoids that on objects sends S_n to $G^n \ltimes \text{Map}(I_{S_n} \rightarrow G \ltimes X)$ and on morphisms sends $d_i : S_n \rightarrow S_{n-1}$ to the functor (d_i^*, σ_i) , formally we have a (covariant) functor

$$\Phi : \quad \mathcal{C}_I^{op} \longrightarrow \mathbf{TrG}$$

$$\begin{array}{ccc} S_n & \longrightarrow & G^n \ltimes \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X) \\ d'_i \uparrow & & \uparrow \sigma_i \ltimes d_i^* \\ S_{n-1} & \longrightarrow & G^{n-1} \ltimes \mathrm{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) \end{array}$$

The *path groupoid* of $G \ltimes X$ is the translation groupoid $P = P(G \ltimes X)$ given by the (filtered) colimit of Φ ,

$$P(G \ltimes X) = \mathrm{colim}_{\mathcal{C}_I^{op}} \Phi$$

where the colim of the functor Φ is an object $P \in \mathbf{TrG}$ together with a morphism from $\mathrm{Map}(I_{S_n} \rightarrow G \ltimes X)$ for each S_n such that for each morphism d_i the following diagram commutes:

$$\begin{array}{ccc} G^n \ltimes \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X) & & \\ \downarrow \sigma_i \ltimes d_i^* & \searrow & \\ G^{n-1} \ltimes \mathrm{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) & \nearrow & P \end{array}$$

Moreover, $P = \mathrm{colim} \Phi$ has the following universal property. Given another translation groupoid W with maps from $G^n \ltimes \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X)$ such maps factor uniquely through the colimit P as shown in the following diagram:

$$\begin{array}{ccc} G^n \ltimes \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X) & & \\ \downarrow \sigma_i \ltimes d_i^* & \searrow & \\ G^{n-1} \ltimes \mathrm{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) & \nearrow & P \end{array} \quad \begin{array}{c} \curvearrowright \\ \text{---} \\ \curvearrowleft \end{array} \quad \begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array} \quad W$$

We are ready now to give an explicit construction of the groupoid $P = P(G \ltimes X)$ by using the constructions of colimits in the category of topological spaces \mathbf{Top} and in the category of groups \mathbf{Grp} .

The colimit of the contravariant functor $\psi : \mathcal{C}_I^{op} \rightarrow \mathbf{Top}$ that on objects sends S_n to $\mathrm{Map}(I_{S_n} \rightarrow G \ltimes X)$ and on morphisms sends $d_i : S_n \rightarrow S_{n-1}$ to the morphism $d_i^* : \mathrm{Map}(I_{S_{n-1}} \rightarrow G \ltimes X) \rightarrow \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X)$ is a topological space $M = \mathrm{colim} \psi$ such that

$$M = \left(\coprod_{\mathcal{C}_I} \mathrm{Map}(I_{S_n} \rightarrow G \ltimes X) \right) / \sim$$

where \sim is the equivalence relation generated by $\alpha \sim d_i^*(\alpha)$ for all S_n and $d_i : S_n \rightarrow S_{n-1}$.

This topological space $M = \text{colim } \psi$ will be the space of objects of the path groupoid P . To construct the space of arrows of the path groupoid, we consider now a colimit in the category of groups.

Consider the functor $\varphi : \mathcal{C}_I^{op} \rightarrow \mathbf{Grp}$ which sends S_n to G^n and on morphisms sends $d_i : S_n \rightarrow S_{n-1}$ to the morphism $\sigma_i : G^{n-1} \rightarrow G^n$ given by $\sigma_i(g_1, \dots, g_{n-1}) = (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$.

The colimit of φ is a group $H = \text{colim } \varphi$ such that

$$H = (\coprod_{\mathcal{C}_I} G^n) / \sim$$

where \sim is generated by $(g_1, \dots, g_{n-1}) \sim (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$. This group H acts on the topological space M constructed above.

We can describe now explicitly the object and arrow spaces of the path groupoid $P = P(G \ltimes X)$ in \mathbf{TrG} :

$$P_0 = M = \text{colim } \psi = \coprod_{\mathcal{C}_I} \text{Map}(I_{S_n} \rightarrow G \ltimes X) / \sim$$

and

$$P_1 = H \times M = \text{colim } \varphi \times \text{colim } \psi = (\coprod_{\mathcal{C}_I} G^n) / \sim \times (\coprod_{\mathcal{C}_I} \text{Map}(I_{S_n} \rightarrow G \ltimes X)) / \sim$$

which we endow with the inductive topology.

Remark 3.5. Let G be a discrete group acting on X . The *path groupoid* of $G \ltimes X$ is the translation groupoid $P = P(G \ltimes X)$ given by the following colimit

$$P = \text{colim}_{\mathcal{C}_I^{op}} G^n \ltimes \text{colim}_{\mathcal{C}_I^{op}} ((X^I)^n \times_{X^{n-1}} G^{n-1})$$

We will show that this path groupoid $P = \text{colim } \Phi$ described above is actually equivalent to the translation groupoid $G \ltimes X^I$. In order to give an explicit characterization of the equivalence of categories, we will introduce some auxiliary groupoids which in turn will relate to the idea introduced in [3] of multiple G -paths.

3.2. Multiple G -paths. We will provide now another description of the path groupoid in terms of equivariant generalized maps. We will see that for each G -path, its equivalence class $[I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X]$ contains a representant in \mathbf{MTrG} of the form:

$$I \xleftarrow{\delta} G \ltimes Y \xrightarrow{\phi} G \ltimes X$$

where $G \ltimes Y$ is a translation groupoid.

Given a G -path $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X$, we will construct a space $Y = Y_\alpha$ such that $G \ltimes Y$ is Morita equivalent to I_{S_n} , and maps $\delta : G \ltimes Y \rightarrow I$ and $\phi : G \ltimes Y \rightarrow G \ltimes X$ such that (δ, ϕ) is 2-isomorphic to the given G -path (ϵ, α) .

3.2.1. Construction of $G \ltimes Y_\alpha$. Let $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$. Consider the product space

$$G \times (I_{S_n})_0 = \{(g, (r, i)) \mid g \in G, (r, i) \in [r_{i-1}, r_i]\}$$

and the following identifications:

$$(g, (r_i, i+1)) \sim (k_i^{-1}g, (r_i, i))$$

where k_i is the image of \tilde{r}_i by α . Now Y_α is defined as the quotient space:

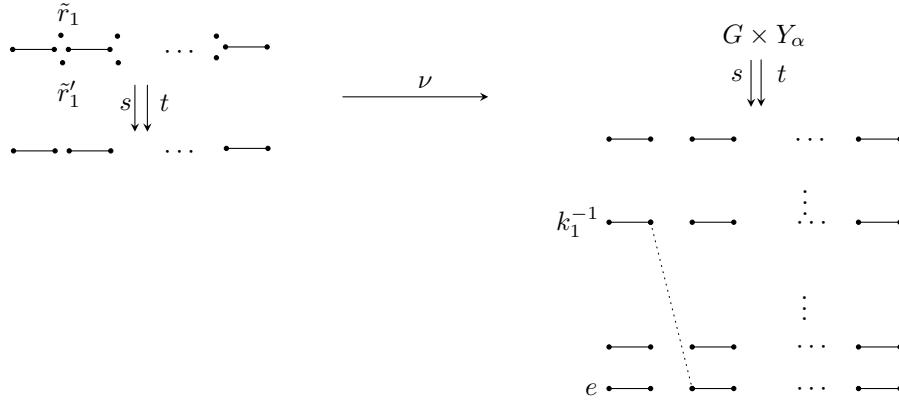
$$Y_\alpha = \{[(g, (r, i))] \mid (g, (r, i)) \in G \times (I_{S_n})_0 \text{ and } (g, (r_i, i+1)) \sim (k_i^{-1}g, (r_i, i))\}.$$

Observe that the space Y_α depends on α in the sense that it is given by the subdivision S_n and the group elements k_1, \dots, k_{n-1} , but it is independent of the actual pieces of the path $\alpha_1, \dots, \alpha_n$.

The action of G on Y_α is given by the multiplication in the group $h([g, (r, i)]) = [gh^{-1}, (r, i)]$.

We can consider then the translation groupoid $G \ltimes Y_\alpha$ where the source and target are given by the maps $s(h, [g, (r, i)]) = [g, (r, i)]$ and $t(h, [g, (r, i)]) = [gh^{-1}, (r, i)]$.

3.2.2. Morita equivalence $I_{S_n} \sim_M G \ltimes Y_\alpha$. We will show now that the translation groupoid constructed above is Morita equivalent to the groupoid I_{S_n} . Let $\nu : I_{S_n} \rightarrow G \ltimes Y_\alpha$ be the morphism defined by $\nu((r, i)) = [e, (r, i)]$ on objects and $\nu(\tilde{r}_i) = (k_i, [e, (r_i, i)])$ on arrows.



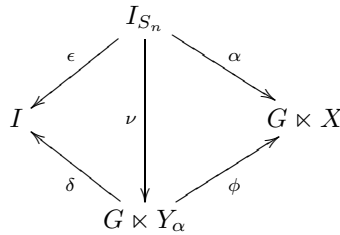
The morphism ν is essentially surjective and fully faithful. Therefore given a groupoid I_{S_n} , we can construct another groupoid Y_α for each set of elements k_1, \dots, k_{n-1} such that I_{S_n} is Morita equivalent to $G \ltimes Y_\alpha$.

3.2.3. The 2-isomorphism $(\epsilon, \alpha) \Rightarrow (\delta, \phi)$. We will define now the maps δ and ϕ to obtain the generalized map $I \xleftarrow{\delta} G \ltimes Y_\alpha \xrightarrow{\phi} G \ltimes X$ being 2-isomorphic to the given G -path (ϵ, α) .

We define $\phi([g, (r, i)]) = g^{-1}\alpha_i(r)$ on objects and $\phi(h, [g, (r, i)]) = (h, g^{-1}\alpha_i(r))$ on arrows. Moreover, the morphism ϕ is G -equivariant in the ordinary sense (the group homomorphism is the identity).

The essential equivalence $\delta : G \ltimes Y_\alpha \rightarrow I$ is given by projection on both objects and arrows, $\delta(h, [g, (r, i)]) = r$. Both morphisms ϕ and δ are well defined and δ is surjective on objects and fully faithful.

We have that the following diagram is commutative:



since $\phi\nu((r, i)) = \phi([e, (r, i)]) = \alpha_i(r_i)$ and $\phi\nu(\tilde{r}_i) = \phi(k_i, [e, (r_i, i)]) = (k_i, \alpha_i(r_i))$.

Therefore there is a 2-isomorphism between $I \xleftarrow{\delta} G \ltimes Y_\alpha \xrightarrow{\phi} G \ltimes X$ and the G -path $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X$.

Observe that the identifications we have made in the quotient to obtain the space Y_α determine a gluing of the small segments $[r_i, r_{i+1}]$ at the different levels of $G \ltimes (I_{S_n})_0$ to obtain copies of the entire interval $I = [0, 1]$. This gluing is determined by the group elements k_1, \dots, k_{n-1} .

To define the map ϕ from the groupoid $G \ltimes Y_\alpha$ associated to the G -path α , we are concatenating the different pieces α_i in these different levels by multiplying by the correct group element to obtain an honest path in X .

3.2.4. The homeomorphism $\gamma : Y_\alpha \rightarrow G \times I$. For each map $\alpha : I_{S_n} \rightarrow G \ltimes X$, let's show now that the space Y_α we just constructed is G -equivariantly homeomorphic to the space $G \times I$, where the action on the latter is determined by the action of G on Y_α given by $h[g, (r, i)] = [gh^{-1}, (r, i)]$. We have that the action on $G \times I$ is given by

$$G \times (G \times I) \rightarrow G \times I$$

$$h, (g, r) = (gh^{-1}, r).$$

We define the homeomorphism $\gamma_\alpha : Y_\alpha \rightarrow G \times I$ as

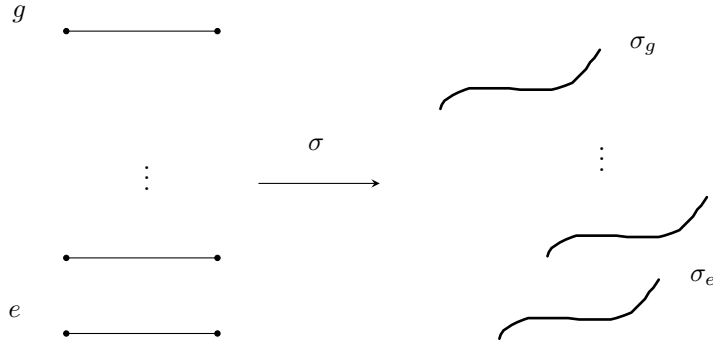
$$\gamma_\alpha([g, (r, i)]) = ((k_{i-1} \cdots k_1)^{-1}g, r)$$

for $i = 1, \dots, n$. The morphism γ_α depends only on S_n and k_1, \dots, k_{n-1} and is independent on the actual paths $\alpha_1, \dots, \alpha_n$. The inverse morphism $\gamma^{-1} : G \times I \rightarrow Y_\alpha$ is given by

$$\gamma^{-1}(h, r) = [k_{i-1} \cdots k_1 h, (r, i)]$$

if $r \in [r_{i-1}, r_i]$. Moreover, we have that the homeomorphism γ_α is G -equivariant by construction.

Definition 3.6. A *multiple G -path* in the groupoid $G \ltimes X$ is a generalized map $I \leftarrow G \ltimes (G \times I) \xrightarrow{\sigma} G \ltimes X$ where σ is a G -equivariant map in the ordinary sense.



3.2.5. *Equivalence of multiple G -paths.* Given two multiple G -paths $I \leftarrow G \ltimes (G \times I) \xrightarrow{\sigma} G \ltimes X$ and $I \leftarrow G \ltimes (G \times I) \xrightarrow{\tau} G \ltimes X$, they are equivalent if there exists a subdivision S_n and $k_1, \dots, k_{n-1} \in G$ such that the following diagram commutes up to natural transformations

$$\begin{array}{ccccc}
 & & G \ltimes (G \times I) & & \\
 & \swarrow p & \uparrow \nu & \searrow \sigma & \\
 I & & I_{S_n} & & G \times X \\
 & \swarrow p & \downarrow \eta & \searrow \tau & \\
 & & G \ltimes (G \times I) & &
 \end{array}$$

where $\nu = \nu_{k_1, \dots, k_{n-1}}$ and $\eta = \eta_{k_1, \dots, k_{n-1}}$.

Since p is an essential equivalence, we have that $\nu \sim \eta$ and then $\sigma\nu \sim \tau\nu$. That means that there exists a natural transformation $T : (I_{S_n})_0 \rightarrow G \times X$ such that $T(r, i)$ is an arrow between $\sigma\nu(r, i) = \sigma((k_{i-1} \cdots k_1)^{-1}, r)$ and $\tau((k_{i-1} \cdots k_1)^{-1}, r)$. Therefore we have that the multiple G -paths are equivalent if there exists a subdivision S_n , $k_1, \dots, k_{n-1} \in G$ and $g_1, \dots, g_n \in G$ such that

$$g_i \sigma((k_{i-1} \cdots k_1)^{-1}, r) = \tau((k_{i-1} \cdots k_1)^{-1}, r) \text{ if } r \in [r_{i-1}, r_i].$$

Since σ is equivariant, we have that

$$g_i(k_{i-1} \cdots k_1) \sigma(e, r) = (k_{i-1} \cdots k_1) \tau(e, r) \text{ if } r \in [r_{i-1}, r_i].$$

For $i = 1$ this means that there exists $g_1 \in G$ such that $\tau(e, r) = g_1 \sigma(e, r)$. Since the interval $e \times I$ is connected, we have that $g_i = (k_{i-1} \cdots k_1) g_1 (k_{i-1} \cdots k_1)^{-1}$ for all $i = 1, \dots, n$. In other words, all other g_i , $i = 2, \dots, n$ are determined by g_1 . Once that we have a group element $g_1 \in G$ that makes $\tau(e, r) = g_1 \sigma(e, r)$ in the first piece of the interval, $r \in [0, r_1]$, then all the other pieces of the interval coming from the subdivision S_n will also coincide since for all $r \in [r_{i-1}, r_i]$

$$g_i(k_{i-1} \cdots k_1) \sigma(e, r) = (k_{i-1} \cdots k_1) \tau(e, r)$$

and

$$g_i = (k_{i-1} \cdots k_1) g_1 (k_{i-1} \cdots k_1)^{-1}$$

Then

$$(k_{i-1} \cdots k_1) g_1 (k_{i-1} \cdots k_1)^{-1} (k_{i-1} \cdots k_1) \sigma(e, r) = (k_{i-1} \cdots k_1) \tau(e, r)$$

which implies that $g_1 \sigma(e, r) = \tau(e, r)$ for all $r \in I$.

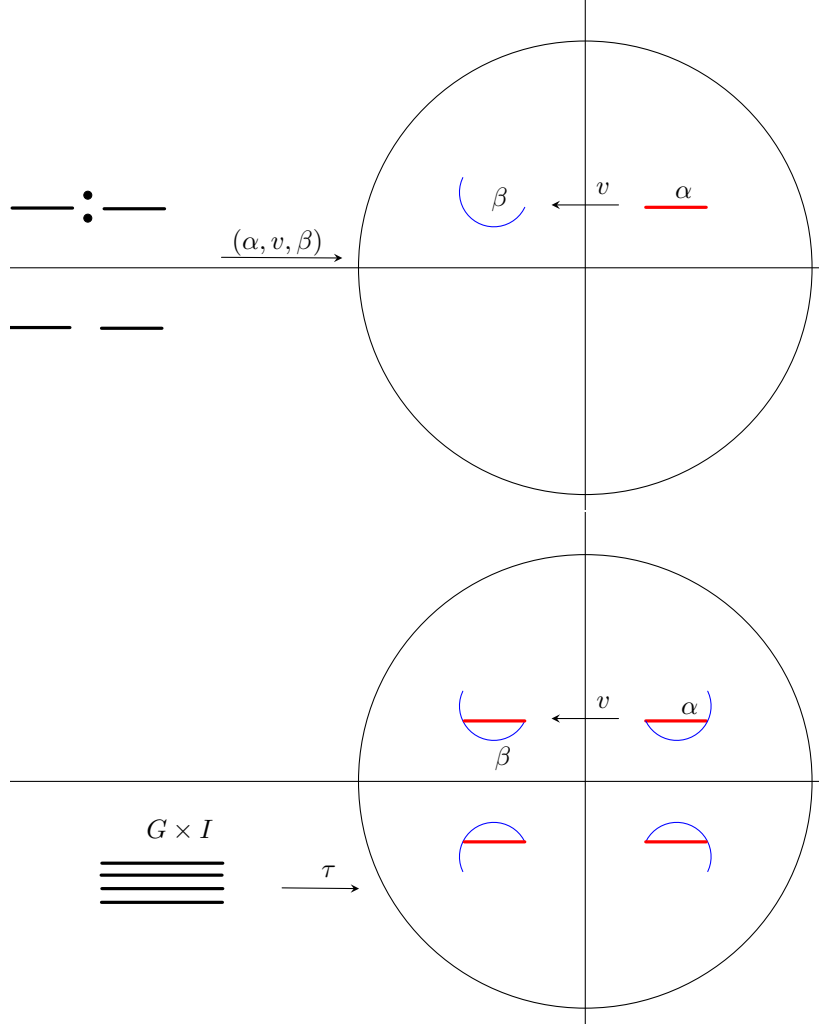
Proposition 3.7. *Two multiple G -paths σ and τ are equivalent if there exists $g \in G$ such that*

$$g \sigma(e, r) = \tau(e, r).$$

Then we have the group G acting now on the space of equivariant maps $G\text{Map}(G \ltimes I, X)$. Let $P' = G \ltimes G\text{Map}(G \ltimes I, X)$ be the multiple path groupoid.

Since $\sigma(g, r) = g \sigma(e, r)$, we observe that a multiple G -path is determined by the honest path $\beta : I \rightarrow X$ given by $\beta(r) = \sigma(e, r)$. Conversely, any path $\beta : I \rightarrow X$ can be made into a multiple G -path by putting $\sigma(g, r) = g \beta(r)$.

Example 3.8. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, u, v, uv\}$ act on the disc $X = D$ by reflection respect to the axes. Starting with the G -path (α, v, β) we construct the associated multiple G -path τ .



We will prove next that all three characterizations of the path groupoid, as G -paths, as multiple G -paths and as honest paths are equivalent.

3.3. Equivalence of the different models for path groupoids. Recall the three definitions of path groupoids that we introduced earlier.

- (1) The groupoid $P = \text{colim } \varphi \ltimes \text{colim } \psi$ where $M = \text{colim } \psi$ is the space of classes of G -paths.
- (2) The groupoid $P' = G \ltimes G\text{Map}(G \times I, X)$ where $G\text{Map}(G \times I, X)$ is the space of G -equivariant maps.
- (3) The groupoid $P'' = G \ltimes X^I$ where X^I is the free path space.

3.3.1. The equivalence of categories $\chi : P = \text{colim } \varphi \ltimes \text{colim } \psi \rightarrow P' = G \ltimes G\text{Map}(G \times I, X)$. Recall that $M = \text{colim } \psi$ is the space of classes of G -paths,

i.e.

$$M = (\coprod_{C_I} \text{Map}(I_{S_n} \rightarrow G \ltimes X)) / \sim$$

where \sim is the equivalence relation generated by $\alpha \sim d_i^*(\alpha)$ for all S_n and $d_i : S_n \rightarrow S_{n-1}$. We will use the same notation $(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ to denote the elements in M .

The idea is to complete each piece α_i of the G -path $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ to have the entire branch σ_i of a multiple G -path σ .

Given a G -path $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ for the subdivision S_n of the interval I we can define (as in the previous section)

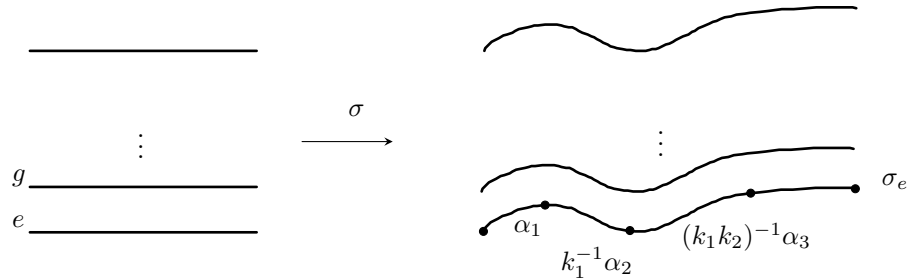
- (1) a space $Y_\alpha = \{[(g, (r, i))] \mid (g, (r, i)) \in G \times (I_{S_n})_0\}$ with the relation $(g, (r_i, i+1)) \sim (k_i^{-1}g, (r_i, i))$,
- (2) a homeomorphism $\gamma_\alpha : G \ltimes Y_\alpha \rightarrow G \ltimes (G \times I)$,
- (3) an essential equivalence $\nu_\alpha : I_{S_n} \rightarrow G \ltimes Y_\alpha$ and
- (4) a generalized map $I \xleftarrow{\delta} G \ltimes Y_\alpha \xrightarrow{\phi_\alpha} G \ltimes X$ such that $(\epsilon, \alpha) \Rightarrow (\delta, \phi_\alpha)$.

We define the functor $\chi : \text{colim } \varphi \ltimes \text{colim } \psi \rightarrow G \ltimes G\text{Map}(G \times I, X)$ as $\chi(\alpha) = \phi_\alpha \gamma_\alpha^{-1}$ on objects and $\chi(g_1, \dots, g_n, \alpha) = (g_1, \alpha)$. Then $\chi(\alpha)(g, r) = \phi_\alpha \gamma_\alpha^{-1}(g, r) = \phi_\alpha[k_{i-1} \dots k_1 g, (r, i)] = (k_{i-1} \dots k_1 g)^{-1} \alpha_i(r)$ if $r \in [r_{i-1}, r_i]$. We are sending each G -path $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ into the multiple G -path σ given by

$$\sigma(g, r) = g^{-1}(k_{i-1} \dots k_1)^{-1} \alpha_i(r) \text{ if } r \in [r_{i-1}, r_i]$$

In particular, we have that the branch σ_e corresponding to the interval $e \times I$ is given by the following concatenation:

$$\sigma(e, r) = \alpha_1(r) * k_1^{-1} \alpha_2(r) * (k_2 k_1)^{-1} \alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1} \alpha_n(r)$$



On arrows, we send $((g_1, \dots, g_n), \alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \in \text{colim } \varphi \times \text{colim } \psi$ into the arrow (g_1, σ_α) where σ_α is defined as before. We can see that χ is an equivariant map between translation groupoids where the group homomorphism is given by the projection on the first coordinate.

Let $\alpha' = (g_1 \alpha_1, \dots, g_n \alpha_n, g_2 k_1 g_1^{-1}, \dots, g_n k_{n-1} g_{n-1}^{-1})$, we have that

$$\chi(\alpha') = g_1 \chi((\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}))$$

since

$$\begin{aligned} \sigma_{\alpha'}(e, r) &= g_1 \alpha_1(r) * (g_2 k_1 g_1^{-1})^{-1} g_2 \alpha_2(r) * \dots * (g_n k_{n-1} g_{n-1}^{-1} \dots g_2 k_1 g_1^{-1})^{-1} g_n \alpha_n(r) \\ &= g_1 (\alpha_1(r) * k_1^{-1} \alpha_2(r) * (k_2 k_1)^{-1} \alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1} \alpha_n(r)) = g_1 \sigma_\alpha(e, r). \end{aligned}$$

Consider the functor given by $\chi^{-1}(\sigma) = \sigma|_{e \times I} \circ i_e$ on objects and $\chi^{-1}((g, \sigma)) = (g, \sigma|_{e \times I} \circ i_e)$, where $i_e : I \rightarrow e \times I$ sends $r \in I$ to $(e, r) \in e \times I$. Recall that by our notation convention the right side means in both cases the class in the colimit. Note that the G -path $\sigma|_{e \times I} \circ i_e$ corresponds to a subdivision S_1 with only one subinterval, that is, $\sigma|_{e \times I} \circ i_e$ is an honest path.

The composition $\chi \circ \chi^{-1} : G \ltimes G\text{Map}(G \times I, X) \rightarrow G \ltimes G\text{Map}(G \times I, X)$ is the identity map since the G -map α_σ associated to σ has only one piece. On objects:

$$\chi \circ \chi^{-1}(\sigma) = \chi(\alpha_\sigma) = \sigma_{\alpha_\sigma}$$

such that $\sigma_{\alpha_\sigma}(g, r) = g^{-1}\sigma(e, r) = \sigma(g, r)$, then $\sigma_{\alpha_\sigma} = \sigma$. On arrows

$$\chi \circ \chi^{-1}(g, \sigma) = \chi(g, \sigma_{\alpha_\sigma}) = \chi(g, \sigma) = (g, \sigma).$$

We will prove next that the composition in the other direction is equivalent by a natural transformation to the identity. We have that $\chi^{-1} \circ \chi : \text{colim } \varphi \ltimes \text{colim } \psi \rightarrow \text{colim } \varphi \ltimes \text{colim } \psi$ sends each G -path class $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ to the G -path α_{σ_α} where

$\alpha_{\sigma_\alpha}(r) = \sigma_\alpha(e, r) = \alpha_1(r) * k_1^{-1}\alpha_2(r) * (k_2k_1)^{-1}\alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1}\alpha_n(r)$ and each arrow $((g_1, \dots, g_n), \alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \in \text{colim } \varphi \times \text{colim } \psi$ to the arrow $(g_1, \alpha_{\sigma_\alpha})$.

There is a natural transformation $T : \text{colim } \psi \rightarrow \text{colim } \varphi \times \text{colim } \psi$ given by

$$T(\alpha) = ((\text{id}, k_1^{-1}, (k_2k_1)^{-1}, \dots, (k_{n-1} \dots k_1)^{-1}), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}))$$

which is an arrow between α and α_{σ_α} since

$$\begin{aligned} & (\text{id}, k_1^{-1}, (k_2k_1)^{-1}, \dots, (k_{n-1} \dots k_1)^{-1})(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) = \\ & ((\text{id } \alpha_1, k_1^{-1}\alpha_2, \dots, (k_{n-1} \dots k_1)^{-1}\alpha_n), (k_1^{-1}k_1, \dots, (k_{n-1} \dots k_1)^{-1}k_{n-1}(k_{n-2} \dots k_1))) \\ & ((\alpha_1, k_1^{-1}\alpha_2, \dots, (k_{n-1} \dots k_1)^{-1}\alpha_n), (\text{id}, \dots, \text{id})). \end{aligned}$$

This G -path is equal to the concatenation of the n pieces $\alpha_1(r) * k_1^{-1}\alpha_2(r) * (k_2k_1)^{-1}\alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1}\alpha_n(r)$ since the connecting arrows are all identities. Moreover, T satisfies the naturality condition.

Therefore χ is an equivalence of categories between the groupoid of G -paths and the groupoid of multiple G -paths. We will see next that the groupoid of multiple G -paths is just the free path space X^I together with G acting on it.

3.3.2. The isomorphism of categories $\xi : P' = G \ltimes G\text{Map}(G \times I, X) \rightarrow P'' = G \ltimes X^I$. To construct this isomorphism we will use the fact that since a multiple G -path σ is equivariant, then it is determined by its value at the branch σ_e corresponding to the interval $e \times I$.

We define $\xi(\sigma) = \sigma i_e \in X^I$ on objects and $\xi(g, \sigma) = (g, \sigma i_e)$ on arrows. Conversely, $\xi^{-1}(\beta) = \sigma_\beta$ where $\sigma_\beta(g, r) = g^{-1}\beta(r)$. The functor $\xi : G \ltimes G\text{Map}(G \times I, X) \rightarrow G \ltimes X^I$ is an isomorphism of categories since it has a strict inverse functor: $\xi \circ \xi^{-1} = \text{id}_{G \ltimes X^I}$ and $\xi^{-1} \circ \xi = \text{id}_{G \ltimes G\text{Map}(G \times I, X)}$.

Theorem 3.9. *All models for the path groupoid of $G \ltimes X$ are equivalent.*

$$P(G \ltimes X) = \text{colim } \varphi \ltimes \text{colim } \psi \sim G \ltimes G\text{Map}(G \times I, X) = G \ltimes X^I.$$

3.4. Functoriality and Morita invariance of the path groupoid. In this section we will see that the path groupoid is functorial and that the path groupoid is well defined up to Morita equivalence.

3.4.1. *Functoriality.*

Definition 3.10. For a strict equivariant map $\varphi \ltimes f : G \ltimes X \rightarrow H \ltimes Y$ we have an induced map $\varphi_* \ltimes f_* : G \ltimes X^I \rightarrow H \ltimes Y^I$ by $f_*(\alpha) = f \circ \alpha$ for all $\alpha \in X^I$ and $\varphi_* = \varphi$.

This map induces an equivariant map $P(\varphi \ltimes f) : P(G \ltimes X) \rightarrow P(H \ltimes Y)$ in the following way, $P(G \ltimes X) = P$ the colimit of $\text{Map}(I_{S_n}, G \ltimes X)$.

For every n we have induced maps

$$(\varphi \ltimes f)_* : \text{Map}(I_{S_n}, G \ltimes X) \rightarrow \text{Map}(I_{S_n}, H \ltimes Y)$$

in terms of the description $\text{Map}(I_{S_n}, G \ltimes X) = G^n \ltimes (X^I)^n \times_{X^{n-1}} G^{n-1}$, this map corresponds just to $\varphi^n \ltimes (f_*^{n-1} \times \varphi^n)$, by taking the colimit we get an equivariant map $P(\varphi \ltimes f) : P(G \ltimes X) \rightarrow P(H \ltimes Y)$.

Similarly, we have an induced map $\varphi_* \ltimes f_* : G \ltimes G\text{Map}(G \times I, X) \rightarrow H \ltimes H\text{Map}(H \times I, Y)$ by taking an equivariant map $(G \times I) \xrightarrow{\sigma} X$ and defining $f_*(\sigma) : H \times I \rightarrow Y$ by $f_*(\sigma)(h, r) = h^{-1}f(\sigma(e, r))$.

In any of the three models the functoriality is easy to check and we have,

Theorem 3.11. *The path groupoid of $G \ltimes X$ is functorial for equivariant maps.*

Moreover, the equivalence of the three models for the path groupoid is natural,

Theorem 3.12. *For a strict equivariant map $\varphi \ltimes f : G \ltimes X \rightarrow H \ltimes Y$ the following diagram is commutative:*

$$\begin{array}{ccc} P(G \ltimes X) & \xrightarrow{P(\varphi \ltimes f)} & P(H \ltimes Y) \\ \downarrow \chi & & \downarrow \chi \\ G \ltimes G\text{Map}(G \times I, X) & \xrightarrow{\varphi_* \ltimes f_*} & H \ltimes H\text{Map}(H \times I, Y) \\ \downarrow \xi & & \downarrow \xi \\ G \ltimes X^I & \xrightarrow{\varphi_* \ltimes f_*} & H \ltimes Y^I \end{array}$$

3.4.2. *Morita Invariance.* Since we have proved that three models of the path groupoid P, P', P'' are equivalent categories we will work with the G -paths and prove that if we take an essential equivalence $G \ltimes X \rightarrow H \ltimes Y$, we have an induced essential equivalence between the G -paths $P(G \ltimes X) \rightarrow P(H \ltimes Y)$.

This will give that for a given Morita equivalence

$$G \ltimes X \xleftarrow{\sigma} G' \ltimes X' \xrightarrow{\epsilon} H \ltimes Y$$

where ϵ and σ are essential equivalences, we have induced essential equivalences

$$P(G \ltimes X) \xleftarrow{P(\sigma)} P(G' \ltimes X') \xrightarrow{P(\epsilon)} P(H \ltimes Y).$$

Now suppose that we have a strict equivariant map $\varphi \ltimes f : G \ltimes X \rightarrow H \ltimes Y$ that is an essential equivalence, let's see that $P(\varphi \ltimes f)$ is an essential equivalence.

Lets see that is fully faithful in the sense that $P(G \ltimes X)_1$ is the following pullback of topological spaces:

$$\begin{array}{ccc} P(G \ltimes X)_1 & \xrightarrow{\epsilon} & P(H \ltimes Y)_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ P(G \ltimes X)_0 \times P(G \ltimes X)_0 & \xrightarrow{\epsilon \times \epsilon} & P(H \ltimes Y)_0 \times P(H \ltimes Y)_0 \end{array}$$

Specifically we have to prove that the natural map from $P(G \ltimes X)_1$ to the fibered product $P(G \ltimes X)_0 \times P(G \ltimes X)_0 \times_{P(H \ltimes Y)_0 \times P(H \ltimes Y)_0} P(H \ltimes Y)_1$ is a homeomorphism.

Let us define the inverse map, for $\alpha, \beta \in P(G \ltimes X)_0$ and an element $\theta \in P(H \ltimes Y)_1$ with $s(\theta) = \alpha, t(\theta) = \beta$ we can assume that there is a subdivision of the interval such that α, β are represented both by elements of $\text{Map}(I_{S_n}, G \ltimes X)$ and θ by an element of H^n . Therefore we have $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ and $\beta = (\beta_1, \dots, \beta_n, k'_1, \dots, k'_{n-1})$ such that $f(\beta_i(r)) = h_i f(\alpha_i(r))$ for all $i = 1, \dots, n$ and $k'_i = h_{i+1} k_i h_i^{-1}$ for all $i = 1, \dots, n-1$.

But then by fixing r and using that $\varphi \ltimes f$ is an essential equivalence we have a fibered product of topological spaces

$$\begin{array}{ccc} (G \ltimes X)_1 & \xrightarrow{\epsilon} & (H \ltimes Y)_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ (G \ltimes X)_0 \times (G \ltimes X)_0 & \xrightarrow{\epsilon \times \epsilon} & (H \ltimes Y)_0 \times (H \ltimes Y)_0 \end{array}$$

and therefore for every $r \in [r_{i-1}, r_i]$ there is $g_i^r \in G$ such that $\phi(g_i^r) = h_i$, but since G is discrete and the dependence on r is continuous, the n -tuple (g_1^r, \dots, g_n^r) actually does not depend on r and represent an element of $P(G \ltimes X)_1$.

Let us see that $P(\varphi \ltimes f)$ is essentially surjective in the sense that

$$s\pi_1 : P(H \ltimes Y)_1 \times_{P(H \ltimes Y)_0}^t P(G \ltimes X)_0 \rightarrow P(H \ltimes Y)_0$$

is an open surjection.

For etale groupoids the condition that the morphism $s\pi_1 : (H \ltimes Y)_1 \times_{(H \ltimes Y)_0}^t (G \ltimes X)_0 \rightarrow (H \ltimes Y)_0$ is an open surjection implies that it has local sections. We will use these local sections to construct local sections of $s\pi_1 : P(H \ltimes Y)_1 \times_{P(H \ltimes Y)_0}^t P(G \ltimes X)_0 \rightarrow P(H \ltimes Y)_0$.

Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a cover of Y and $s_\alpha : U_\alpha \rightarrow (H \ltimes Y)_1 \times_{(H \ltimes Y)_0}^t (G \ltimes X)_0$ the local sections. Take $\gamma \in P(H \ltimes Y)_0$ and suppose that γ is represented by an element of $\text{Map}(I_{S_n}, H \ltimes Y)$ therefore $\gamma = (\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1})$ with $\gamma_i : [r_{i-1}, r_i] \rightarrow Y$.

By compactness of the interval $[r_{i-1}, r_i]$ we can find a partition $r_{i-1} = s_0^i \leq \dots \leq s_{m_i}^i = r_i$ such that each $\gamma_i([s_{j-1}^i, s_j^i])$ is contained in some $U_{\alpha_j^i}$.

With the local sections $s_{\alpha_j^i}$ we obtain paths $\pi_2 s_{\alpha_j^i}(\gamma_i(r)) : [s_{j-1}^i, s_j^i] \rightarrow X$ and functions $\pi_1 \pi_2 s_{\alpha_j^i}(\gamma_i(r)) : [s_{j-1}^i, s_j^i] \rightarrow H$, since the intervals are connected and H is a discrete group, actually these functions are constant and we have elements $h_j^i \in H$ with $s\pi_1(f(\pi_2 s_{\alpha_j^i}(\gamma_i(r))), h_j^i) = \gamma_i(r)$, i.e.

$$f(\pi_2 s_{\alpha_j^i}(\gamma_i(r))) = h_j^i \gamma_i(r)$$

for $r \in [s_{j-1}^i, s_j^i]$.

Note that $(h_j^i)^{-1} f(\pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i))) = (h_{j+1}^i)^{-1} f(\pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i)))$ (both are $\gamma_i(s_j^i)$) and therefore

$$h_{j+1}^i (h_j^i)^{-1} f(\pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i))) = f(\pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i))).$$

Since f is full and faithful, there is a $g_j^i \in G$ with $\phi(g_j^i) = h_{j+1}^i (h_j^i)^{-1}$, such that

$$g_j^i \pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i)) = \pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i))$$

Similarly at the end points of two consecutive paths we have $k_i \gamma_i(r_i) = \gamma_{i+1}(r_i)$ and therefore

$$k_i (h_{m_i}^i)^{-1} f(\pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i))) = k_i \gamma_i(r_i) = \gamma_{i+1}(r_i) = (h_0^{i+1})^{-1} f(\pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i)))$$

so

$$h_0^{i+1} k_i (h_{m_i}^i)^{-1} f(\pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i))) = f(\pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i)))$$

and since we have that f is full and faithful then, we have a elements $g^i \in G$ with $\phi(g^i) = h_0^{i+1} k_i (h_{m_i}^i)^{-1}$, such that

$$g^i \pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i)) = \pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i)).$$

Therefore we have a G -path $\left(\left(\pi_2 s_{\alpha_j^i}(\gamma_i(r)) \right)_{i,j}, g_1^1, g_2^1, \dots, g_{m_1}^1, g_1^2, g_2^2, \dots, g_{m_n}^n \right)$ and elements $(h_1^1, h_2^1, \dots, h_{m_1-1}^1, \dots, h_{m_n-1}^n)$ of H . By construction,

$$(h_1^1, h_2^1, \dots, h_{m_1-1}^1, \dots, h_{m_n-1}^n) \left(\left(\pi_2 s_{\alpha_j^i}(\gamma_i(r)) \right)_{i,j}, g_1^1, g_2^1, \dots, g_{m_1}^1, g_1^2, g_2^2, \dots, g_{m_n}^n \right)$$

is

$$(\gamma_1|_{[s_0^1, s_1^1]}, \gamma_1|_{[s_1^1, s_2^1]} \dots, \gamma_1|_{[s_{m_1-1}^1, s_{m_1}^1]}, \dots, \gamma_n|_{[s_{m_n-1}^n, s_{m_n}^n]}, id, id, \dots, k_1, id, \dots, k_n).$$

In the colimit this represent the same element as $(\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1})$. Therefore we have constructed local sections on the set

$$\{(\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1}) \in \text{Map}(I_{S_n}, H \ltimes Y) \mid \gamma_i([s_{j-1}^i, s_j^i]) \subseteq U_{\alpha_j^i}\}$$

which is an open set in the compact open topology of $\text{Map}(I_{S_n}, H \ltimes Y)$ (is the intersection of the elements of the subbasis that send $[s_{j-1}^i, s_j^i]$ into $U_{\alpha_j^i}$).

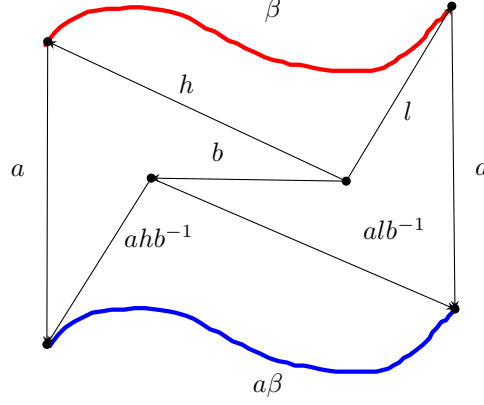
Thus, we have proved

Theorem 3.13. *The path groupoid functor sends essential equivalences to essential equivalences and therefore is well defined up to Morita equivalence.*

3.5. The free loop groupoid $L(G \ltimes X)$. From this model of the path groupoid, we define the loop groupoid as the following pullback:

$$\begin{array}{ccc} & G \ltimes X^I & \\ & \downarrow \text{ev} & \\ \Delta : G \ltimes X & \longrightarrow & (G \times G) \ltimes (X \times X) \end{array}$$

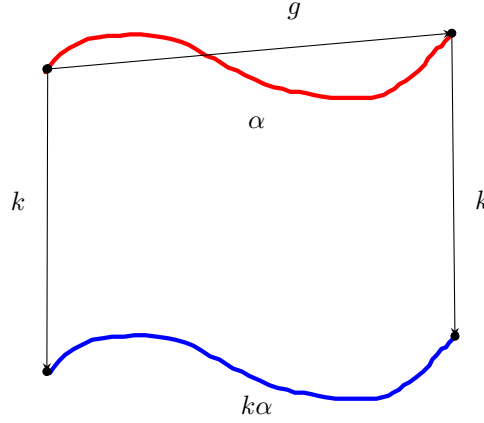
The space of objects is $L_0 = \{(\beta, h, l, x) \in X^I \times G \times G \times X \mid \beta(0) = hx \text{ and } \beta(1) = lx\} = \{(\beta, h, l) \in X^I \times G \times G \mid \beta(0) = hl^{-1}\beta(1)\}$ and the group $G \times G$ acts on L_0 by $(a, b)(\beta, h, l) = (a\beta, bha^{-1}, bla^{-1})$. The following diagram depicts an arrow $(a, b) \in G \times G$ from (β, h, l) to $(a\beta, bha^{-1}, bla^{-1})$.



If we take $b = ah$, we have that an object (β, h, l) is equivalent to a path of the form (α, e, g) , then we can rewrite the space of objects of the loop groupoid as

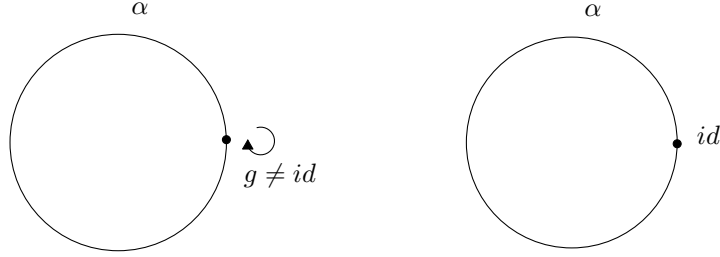
$$L = \{(\alpha, g) \in X^I \times G \mid \alpha(0) = g\alpha(1)\}$$

and $(\alpha, g) \sim (k\alpha, kgk^{-1})$ since $(\alpha, g) \sim (k\alpha, (bhk^{-1})^{-1}blk^{-1}) = (k\alpha, kh^{-1}lk^{-1}) = (k\alpha, kgk^{-1})$. The following diagram depicts an arrow $(k, (\alpha, g))$.



Then, the loop groupoid $(G \times G) \ltimes L_0$ is equivalent to the translation groupoid $G \ltimes L$.

Notice that we no longer have the familiar injection of loops into paths that we have in the category of topological spaces. This aspect of our theory exhibits an important departure from the homotopy theory of topological spaces. In particular, there are non-equivalent G -loops that become equivalent when considered as G -paths. See for instance the action of \mathbb{Z}_2 in S^1 as in Example 3.4. Consider the following G -loops (α, g) and $\alpha \in L(\mathbb{Z}_2 \ltimes S^1)$



We have that $(\alpha, g) \sim \alpha$ as free paths but $(\alpha, g) \not\sim \alpha$ as free loops. Indeed, $(\alpha, g, c) \sim \alpha * g^{-1}c \sim \alpha$ as G -paths but there is no arrow in $L(G \ltimes X)$ between (α, g) and α since $\alpha(t)$ is not a fixed point of the action for all $t \in I$. The map in the groupoid pullback defining the loop groupoid is the projection on objects:

$$L(G \ltimes X) \longrightarrow P(G \ltimes X)$$

$$(\alpha, g) \xrightarrow{i} \alpha$$

which in particular is not injective.

Using these descriptions for the path groupoid in the special case of the point groupoid, we have that the free loop groupoid of $G \ltimes \bullet$ is $(G \times G) \ltimes (G \times G)$ with the action $(a, b) \cdot (h, l) = (bha^{-1}, bla^{-1})$. This groupoid is equivalent to G acting on itself by conjugation by using the second characterization of the loop groupoid as $G \ltimes L$ with $L = \{(\beta, g) \in X^I \times G \mid \beta(0) = g\beta(1)\}$. In this way, we recover a result of Lupercio and Uribe in [7]. Observe that $L(G \ltimes \bullet) = G \ltimes G$ whereas $P(G \ltimes \bullet) = G \ltimes \bullet$.

3.6. Based path groupoids. Once that we have defined the free path groupoid of a translation groupoid and have given several equivalent models, we can give an explicit characterization of the various groupoids resulting of fixing some points. These based groupoids of paths will be of great significance to the groupoid Lusternik-Schnirelman theory defined in [2] as well as for the groupoid topological complexity defined in [4] and they will be discussed elsewhere.

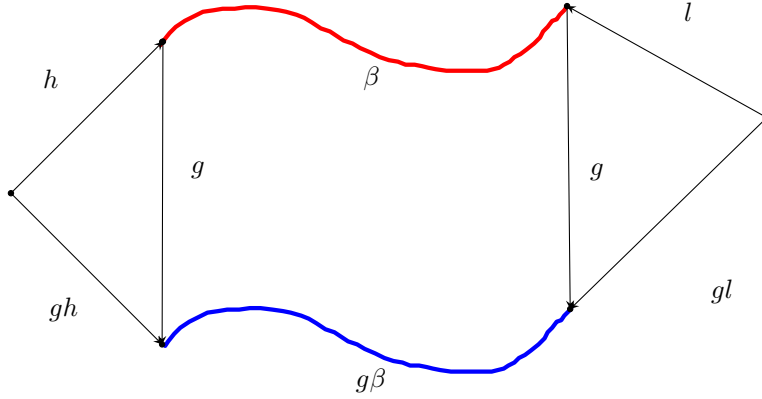
3.6.1. The groupoid $\Omega_{x,y}$ of paths from x to y . The groupoid of paths from x to y , $\Omega_{x,y}$, is defined as a pullback of the evaluation map $\text{ev} : P(G \ltimes X) \rightarrow (G \times G) \ltimes (X \times X)$ and the constant map $x \times y : \mathbf{1} \rightarrow (G \times G) \ltimes (X \times X)$ where $\mathbf{1}$ is the trivial groupoid with one object and one arrow, $\mathbf{1} = e \ltimes \bullet$ and $(x \times y)(\bullet) = (x, y)$. That is,

$$\begin{array}{ccc} \Omega_{x,y} & \longrightarrow & P(G \ltimes X) \\ \downarrow & & \downarrow \text{ev} \\ \mathbf{1} & \xrightarrow{x \times y} & (G \times G) \ltimes (X \times X) \end{array} .$$

Note that by the definition of groupoid pullback, we have that if we take the model of the path groupoid of G -paths, $P = \text{colim } \phi \ltimes \text{colim } \psi$, then the object space of the pullback is:

$$\{((\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}), h, l) \in \text{colim } \psi \times (G \times G) \mid \alpha_1(0) = hx \text{ and } \alpha_n(1) = ly\}$$

These are paths that start at any point of the orbit of x and end at any point of the orbit of y . The space of arrows is the cartesian product $G \times (\Omega_{x,y})_0$ where the action is given by $g(\beta, h, l) = (g\beta, gh, gl)$.


$$(\alpha, k) \sim (h^{-1}\beta, e, h^{-1}l) \sim (gh^{-1}\beta, ge, gh^{-1}l) = (g\alpha, g, gk) \sim (e\alpha, e, k) \sim (\alpha, k)$$

Consider the two equivalent free G -paths given in part 2 of Example 3.4. We have that (α, g) and α are in $\Omega_{-1,1}$ but they are not equivalent as based paths. Therefore, $(\alpha, g) \sim \alpha$ as free G -paths but $(\alpha, g) \not\sim \alpha$ as paths between x and y . Again, we no longer have the analogy with topological spaces where there is an injection from the based path space to the the free one.

3.6.2. *The groupoid Ω_x of based loops.* Similarly, we define the based loop groupoid as the groupoid pullback,

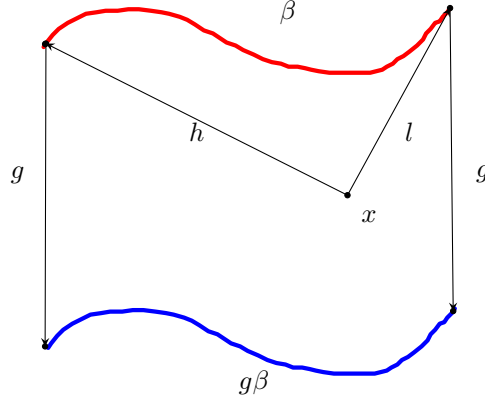
$$\begin{array}{ccc} \Omega_x & \longrightarrow & P(G \ltimes X) \\ \downarrow & & \downarrow \text{ev} \\ \mathbf{1} & \xrightarrow{x \times x} & G \times G \ltimes (X \times X) \end{array}$$

where $x \times x$ is the constant map with $(x \times x)(\bullet) = (x, x)$.

That is, the *based loop groupoid* is the translation groupoid $\Omega_x = G \ltimes (\Omega_x)_0$ where the object space is

$$(\Omega_x)_0 = \{(\beta, h, l) \in X^I \times (G \times G) \mid \beta(0) = hx \text{ and } \beta(1) = lx\}$$

i.e. the space of paths that begin and end at (possibly different) points in the orbit of x . The action is given by $g(\beta, h, l) = (g\beta, gh, gl)$.



Again, the groupoid Ω_x is equivalent to the topological space $P_{x,x} = \{(\alpha, k) \mid \alpha(0) = x \text{ and } \alpha(1) = kx\}$.

Alternatively, the based loop groupoid Ω_x can be obtained as the following groupoid pullback:

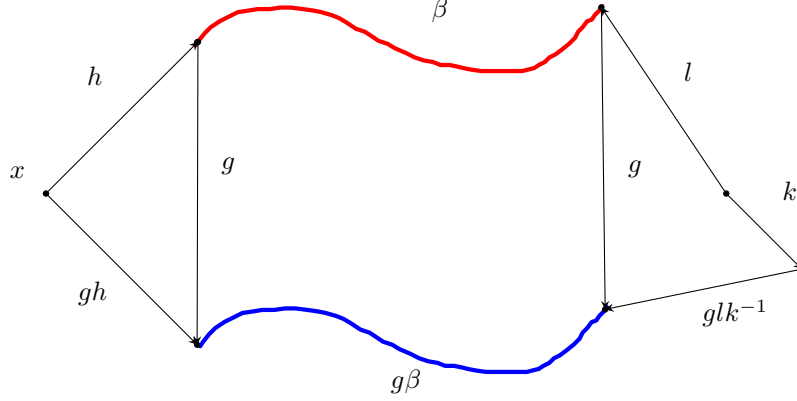
$$\begin{array}{ccc} \Omega_x & \longrightarrow & L(G \ltimes X) \\ \downarrow & & \downarrow \text{ev}_0 \\ \mathbf{1} & \xrightarrow{x} & G \ltimes X \end{array}$$

where $L(G \ltimes X)$ is the free loop groupoid.

3.6.3. *The groupoid P_x of paths from x .* We define the based path groupoid as the following groupoid pullback:

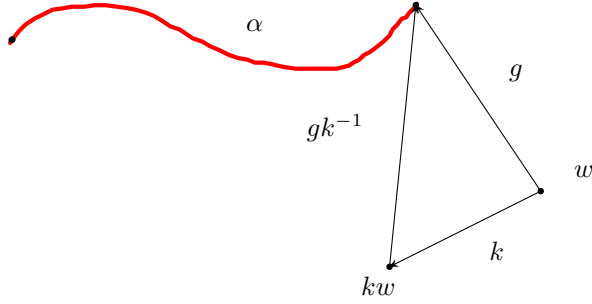
$$\begin{array}{ccc} P_x & \longrightarrow & P(G \ltimes X) \\ \downarrow & & \downarrow \text{ev} \\ \mathbf{1} \times (G \ltimes X) & \xrightarrow{(x, \text{id})} & (G \times G) \ltimes (X \times X) \end{array}$$

where $(x, \text{id}) : \mathbf{1} \times (G \ltimes X) \rightarrow (G \times G) \ltimes (X \times X)$ is given by $(x, \text{id})(\bullet, z) = (x, z)$. Then the object space of the pullback P_x is $(P_x)_0 = \{(\beta, (h, l), (\bullet, z)) \in X^I \times G \times G \times \mathbf{1} \times X \mid \beta(0) = hx \text{ and } \beta(1) = lz\} = \{(\beta, (h, l), z) \mid \beta(0) = hx \text{ and } \beta(1) = lz\}$. The group $G \times G$ acts on $(P_x)_0$ by $(g, k)(\beta, (h, l), z) = (g\beta, (gh, glk^{-1}), kz)$.



The *based path groupoid* is the translation groupoid $P_x = (G \times G) \ltimes (P_x)_0$.

Observing that the equivalence class of each $(\beta, (h, l), z) \in (P_x)_0$ contains an element of the form (α, g, w) we have that the based path groupoid P_x is equivalent to $G \ltimes P$ where $P = \{(\alpha, g, w) \mid \alpha(0) = x \text{ and } \alpha(1) = gw\}$ and the action is given by $k(\alpha, g, w) = (\alpha, gk^{-1}, kw)$. The following diagram depicts an arrow $(k, (\alpha, g, w)) \in G \times P$.



3.7. Examples. We will illustrate the concepts described above by calculating various path groupoids in some particular cases.

3.7.1. Topological spaces. The free path groupoid of the topological space X is $P(e \ltimes X) = e \ltimes X^I = X^I$ and the free loop groupoid is $L(e \ltimes X) = e \ltimes L$ where $L = \{\alpha \in X^I \mid \alpha(0) = \alpha(1)\}$. In this way we recover the classical free path and free loop space of a topological space.

3.7.2. Groups. For a point groupoid $G \ltimes \bullet$ we have showed before that the path groupoid is itself and the loop groupoid is $(G \times G) \ltimes (G \times G)$ with the action $(a, b) \cdot (h, l) = (bha^{-1}, bla^{-1})$ which is equivalent to G acting on itself by conjugation, that is, $L(G \ltimes \bullet) = G \ltimes G$ and $P(G \ltimes \bullet) = G \ltimes \bullet$. The based loop groupoid is the unit groupoid G , as a discrete topological space. The based path groupoid of paths emanating from \bullet is $G \ltimes G$.

3.7.3. Free actions. If G acts freely on a topological space X , we observe that the groupoid $G \ltimes X$ and the topological space X/G are Morita equivalent. Then, we have that $P(G \ltimes X) = P(e \ltimes X/G) = e \ltimes (X/G)^I = (X/G)^I$ and the free loop groupoid is $L(G \ltimes X) = L(X/G)$ where $L(X/G)$ is the free loop space of the topological space X/G . In the same way, we have that the based groupoids coincide with the ones of the topological space X/G .

3.7.4. Orbifolds. In general, for translation orbifolds $G \ltimes X$ we have that the path groupoid is $P(G \ltimes X) = G \ltimes X^I$ and the loop groupoid is $L(G \ltimes X) = G \ltimes L$ where $L = \{(\alpha, g) \in X^I \times G \mid \alpha(0) = g\alpha(1)\}$ is the pullback of the topological spaces $G \times X$ and X^I , i.e. $L = X^I \times_{X \times X} (G \times X) = X^I \times_X G$.

4. HOMOTOPY

We will define in this section a notion of homotopy based on the explicit description of the path groupoid $P(G \ltimes X)$ given in the previous section. This will provide a concrete alternative to the more abstract presentation given by Noohi in [10, 11].

4.1. Natural transformations for translation groupoids. The equivariant maps $\varphi \ltimes f : K \ltimes Z \rightarrow G \ltimes X$ and $\psi \ltimes g : K \ltimes Z \rightarrow G \ltimes X$ are equivalent by a natural transformation if there exists a K -map $\gamma : Z \rightarrow G$ such that $\gamma(z)f(z) = g(z)$ for all $z \in Z$ where both Z and G are K -spaces considering the following action of K on G :

$$K \times G \rightarrow G \text{ where } (k, g) \mapsto \psi(k)g\varphi(k)^{-1}.$$

Therefore $\varphi \ltimes f \sim \psi \ltimes g$ if there exists $\gamma : Z \rightarrow G$ such that

- (1) $\gamma(z)f(z) = g(z)$ for all $z \in Z$ and
- (2) $\gamma(kz) = \psi(k)\gamma(z)\varphi(k)^{-1}$ for all $k \in K$.

If Z is connected, then γ is a constant map since G is discrete. Then $\varphi \ltimes f \sim \psi \ltimes g$ if there exists $h \in G$ such that $hf(z) = g(z)$ for all $z \in Z$ and $h = \psi(k)h\varphi(k)^{-1}$ for all $k \in K$. Then $g = hf$ and $\psi = h^{-1}\varphi h$. In other words, $\psi(k)$ is conjugated of $\varphi(k)$ for all $k \in K$.

If G is abelian, then $\varphi \ltimes f \sim \psi \ltimes g$ if $g = hf$ for some $h \in G$ and $\varphi = \psi$.

If $X = Z = \bullet$, then $\varphi \ltimes \bullet \sim \psi \ltimes \bullet$ iff φ and ψ are conjugate, $\varphi = h^{-1}\psi h$. In particular, when the group acting is abelian we have that two maps between point groupoids are equivalent by a natural transformation only if they are equal.

We give now a characterization of 2-isomorphism for strict maps. Namely if two strict maps are 2-isomorphic then when composed with an essential equivalence they are equivalent by a natural transformation, and if two strict maps are equivalent by a natural transformation then they are 2-isomorphic as generalized maps.

Proposition 4.1. *If f and g are equivalent by a natural transformation, then $f \Rightarrow g$ as generalized equivariant maps.*

Proof. Just consider the essential equivalences η and ν as identity maps and the following diagram is commutative up to natural transformations since $f \sim g$

$$\begin{array}{ccccc}
 & G \ltimes X & & & \\
 \text{id} \swarrow & & \uparrow \text{id} & & \searrow f \\
 G \ltimes X & \sim & G \ltimes X & \sim & H \ltimes Y \\
 \nwarrow \text{id} & & \downarrow \text{id} & & \nearrow g \\
 & G \ltimes X & & &
 \end{array}$$

□

Proposition 4.2. *If two strict maps $f : G \ltimes X \rightarrow H \ltimes Y$ and $g : G \ltimes X \rightarrow H \ltimes Y$ are 2-isomorphic, then there exists an essential equivalence $\eta : \mathcal{L} \rightarrow G \ltimes X$ such that $f\eta \sim g\eta$.*

Proof. We have that there exist essential equivalences η, ν such that the following diagram

$$\begin{array}{ccccc}
 & G \ltimes X & & & \\
 \text{id} \swarrow & & \uparrow \eta & & \searrow f \\
 G \ltimes X & \sim & \mathcal{L} & \sim & H \ltimes Y \\
 \nwarrow \text{id} & & \downarrow \nu & & \nearrow g \\
 & \ell & & &
 \end{array}$$

commutes up to natural transformation. That is $\eta \sim \nu$ and $f\eta \sim g\nu$. Therefore, $f\eta \sim g\eta$. □

Proposition 4.3. *If $(\epsilon, f) \Rightarrow (\sigma, g)$, then there exist essential equivalences ν and η such that $f\nu \Rightarrow g\eta$.*

Proof. By definition of 2-isomorphism, there are essential equivalences ν and η such that $f\nu \sim g\eta$. The result follows from Proposition 4.1. □

Proposition 4.4. *If $f \Rightarrow g$, then $(\epsilon, f) \Rightarrow (\sigma, g)$ for all essential equivalences ϵ, σ with $\epsilon \sim \sigma$.*

4.2. Diagonal map. We will consider the pullback of the unique morphism $G \ltimes X \xrightarrow{\epsilon} \mathbf{1}$ with itself, where $\mathbf{1}$ is the terminal object in \mathbf{MTopG} . This pullback defines the product and then by the universal property we obtain the definition of the diagonal map. Then, the path groupoid will be a factorization of that diagonal.

Definition 4.5. [5] An object T in a bicategory \mathbf{B} is terminal if the category $\mathbf{B}[C, T]$ is equivalent to the terminal category for every object C in \mathbf{B} . A terminal object is unique up to equivalence when it exists.

The trivial groupoid $\mathbf{1} = e \ltimes \bullet$ is the terminal object in the bicategory of translation groupoids \mathbf{MTrG} since the category $\mathbf{MTrG}[G \ltimes X, \mathbf{1}]$ is equivalent to the category $\mathbf{1}$. Indeed, the objects in the category $\mathbf{MTrG}[G \ltimes X, \mathbf{1}]$ are generalized maps and the arrows are classes of diagrams. We can see that all objects are related by an arrow, i.e. $\mathbf{MTrG}[G \ltimes X, \mathbf{1}]$ is the pair groupoid. Given two generalized maps

$G \ltimes X \xleftarrow{\epsilon'} G' \ltimes X' \xrightarrow{c'} \mathbf{1}$ and $G \ltimes X \xleftarrow{\epsilon''} G'' \ltimes X'' \xrightarrow{c''} \mathbf{1}$ we can see that they are equivalent:

$$\begin{array}{ccccc}
 & & G' \ltimes X' & & \\
 & \swarrow \epsilon' & \uparrow & \searrow & \\
 G \ltimes X & \sim & P & \sim & \mathbf{1} \\
 & \swarrow \epsilon'' & \downarrow & \searrow & \\
 & & G'' \ltimes X'' & &
 \end{array}$$

by considering P as the pullback of ϵ' and ϵ'' . In particular, the strict constant map $G \ltimes X \xrightarrow{c} \mathbf{1}$ is the (unique up to 2-iso) map to the terminal object.

Let's now consider the pullback of this constant map with itself which defines the product:

$$\begin{array}{ccc}
 G \times G \ltimes (X \times X) & \longrightarrow & G \ltimes X \\
 \downarrow & & \downarrow c \\
 G \ltimes X & \xrightarrow{c} & \mathbf{1}
 \end{array}$$

The product $(G \times G) \ltimes (X \times X)$ of the object $G \ltimes X$ by itself is unique up to equivalence.

By the universal property of the pullback, we have that there exists a map Δ that makes the two triangles commutative up to natural transformation:

$$\begin{array}{ccccc}
 G \ltimes X & & & & \\
 \swarrow \text{id} & \Delta & \searrow & & \\
 & (G \times G) \ltimes (X \times X) & \xrightarrow{p_1} & G \ltimes X & \\
 \searrow \text{id} & \downarrow p_2 & & \downarrow c & \\
 & G \ltimes X & \xrightarrow{c} & \mathbf{1} &
 \end{array}$$

The map $\Delta : G \ltimes X \rightarrow (G \times G) \ltimes (X \times X)$ is the *diagonal map*. Its explicit definition on objects is $\Delta(x) = (x, x)$ and on arrows, $\Delta(g, x) = (g, g, x, x)$. The diagonal map is defined up to 2-isomorphism.

Remark 4.6. Note that the diagonal defined in [1] is 2-isomorphic to this one.

Definition 4.7. The *evaluation map* $\text{ev} : G \ltimes X^I \rightarrow (G \times G) \ltimes (X \times X)$ is given by $\text{ev}(g, \alpha) = (g, g, \alpha(0), \alpha(1))$.

We have that the diagonal map factors through the path groupoid as expected.

Proposition 4.8. *There is a factorization of the diagonal map Δ*

$$\begin{array}{ccc}
 G \ltimes X & \xrightarrow{\Delta} & (G \times G) \ltimes (X \times X) \\
 \searrow k & & \nearrow e \\
 & G \ltimes X^I &
 \end{array}$$

where k and e are generalized maps.

Proof. Let k be the functor $G \ltimes X \rightarrow G \ltimes X^I$ given by $x \rightsquigarrow \alpha_x$ on objects, and $(g, x) \rightsquigarrow (g, \alpha_x)$ where $\alpha_x : I \rightarrow X$ is a constant path at $x \in X$ and let e be the evaluation map, $e = \text{ev}$. Then, we have that the composition $e \circ c$ is equivalent by a natural transformation to the diagonal Δ . \square

4.3. Homotopic maps. We will give now an explicit characterization of the homotopy between generalized maps.

Definition 4.9. Two generalized maps $K \ltimes Y \xleftarrow{\sigma} K' \ltimes Y' \xrightarrow{f} G \ltimes X$ and $K \ltimes Y \xleftarrow{\tau} K'' \ltimes Y'' \xrightarrow{g} G \ltimes X$ are *homotopic* if there is a generalized map $K \ltimes Y \xleftarrow{\epsilon} \tilde{K} \ltimes \tilde{Y} \xrightarrow{H} G \ltimes X$ such that the following diagram commutes up to 2-isomorphism

$$\begin{array}{ccccc}
 G \ltimes X & \xleftarrow{\text{ev}_0} & G \ltimes X^I & \xrightarrow{\text{ev}_1} & G \ltimes X \\
 & \searrow f & \uparrow H & & \nearrow g \\
 & K' \ltimes Y' & \tilde{K} \ltimes \tilde{Y} & & K'' \ltimes Y'' \\
 & \searrow \sigma & \downarrow \epsilon & \swarrow \tau & \\
 & & K \ltimes Y & &
 \end{array}$$

This means that the generalized map (σ, f) is isomorphic to the generalized map $(\epsilon, \text{ev}_0 \circ H)$ and (τ, g) is isomorphic to $(\epsilon, \text{ev}_1 \circ H)$.

That is (σ, f) is homotopic to (τ, g) if there exists (ϵ, H) and two commutative diagrams up to natural transformations:

$$\begin{array}{ccccc}
 & & \tilde{K} \ltimes \tilde{Y} & & \\
 & \swarrow \epsilon & \uparrow u_0 & \searrow \text{ev}_0 H & \\
 \mathcal{K} \ltimes Y & \sim & \mathcal{L}_0 & \sim & G \ltimes X \\
 & \swarrow \sigma & \downarrow v_0 & \searrow f & \\
 & & K' \ltimes Y' & &
 \end{array}
 \quad
 \begin{array}{ccccc}
 & & \tilde{K} \ltimes \tilde{Y} & & \\
 & \swarrow \epsilon & \uparrow u_1 & \searrow \text{ev}_1 H & \\
 \mathcal{K} \ltimes Y & \sim & \mathcal{L}_1 & \sim & G \ltimes X \\
 & \swarrow \tau & \downarrow v_1 & \searrow g & \\
 & & K'' \ltimes Y'' & &
 \end{array}$$

where \mathcal{L}_i is a translation groupoid, and u_i and v_i are equivariant essential equivalences for $i = 0, 1$. We will denote this homotopy between equivariant generalized maps by \simeq .

Remark 4.10. $(\sigma, f) \simeq (\tau, g)$ if $\exists(\epsilon, H), u_0, u_1, v_0, v_1$ such that

$$f v_0 \sim \text{ev}_0 H u_0 \text{ and } g v_1 \sim \text{ev}_1 H u_1$$

with $\epsilon u_0 \sim \sigma v_0$ and $\epsilon u_1 \sim \tau v_1$.

Proposition 4.11. *If $(\sigma, f) \Rightarrow (\tau, g)$, then $(\sigma, f) \simeq (\tau, g)$.*

Proof. Consider $H = i_X \circ f$ where i_X is the inclusion of X in X^I given by $i_X(x) = \alpha_x$ with α_x being the constant map $\alpha_x(t) = x$ for all $t \in I$. Then the following

diagram is commutative up to 2-isomorphism:

$$\begin{array}{ccccc}
 G \ltimes X & \xleftarrow{\text{ev}_0} & G \ltimes X^I & \xrightarrow{\text{ev}_1} & G \ltimes X \\
 & \searrow f & \uparrow H & & \nearrow g \\
 & K' \ltimes Y' & K' \ltimes Y' & & K'' \ltimes Y'' \\
 & & \downarrow \sigma & \swarrow \tau & \\
 & & K \ltimes Y & &
 \end{array}$$

The first triangle is an equality and the second one is commutative since $(\sigma, f) \Rightarrow (\tau, g)$. \square

Remark 4.12. Let f and g be strict maps. Following the characterization for isomorphism of strict maps given in Proposition 4.2, we have that $f \simeq g$ if there exist essential equivalences η and ν such that $f\sigma\eta \sim \text{ev}_0 H\eta$ and $g\sigma\nu \sim \text{ev}_1 H\nu$.

Proposition 4.13. *Let f and g be strict maps.*

- (1) *If f and g are ψ -equivariantly homotopic maps, then $f \simeq g$ as generalized equivariant maps.*
- (2) *If f and g are equivalent by a natural transformation, then $f \simeq g$ as generalized equivariant maps.*

Proof. (1) Let $H : Y \rightarrow X^I$ be the ψ -equivariant homotopy, i.e. $H_t(ky) = \psi(k)H_t(y)$. Then the following diagram is commutative:

$$\begin{array}{ccccc}
 G \ltimes X & \xleftarrow{\text{ev}_0} & G \ltimes X^I & \xrightarrow{\text{ev}_1} & G \ltimes X \\
 & \searrow f & \uparrow H & & \nearrow g \\
 & & K \ltimes Y & &
 \end{array}$$

- (2) Follows from Proposition 4.1 and Corollary 4.11. \square

Therefore our definition of homotopy generalizes both the notion of natural transformation and the notion of equivariant homotopy.

Proposition 4.14. *If $(\epsilon, f) \simeq (\sigma, g)$ then there exist essential equivalences a and b such that $fa \simeq gb$ as strict maps.*

Proof. Since we have a homotopy between generalized maps, we know that there exists (δ, H) and essential equivalences u_0, v_0, u_1, v_1 such that $f v_0 \sim \text{ev}_0 H u_0, g v_1 \sim \text{ev}_1 H u_1$ and $\delta u_0 \sim \text{ev}_0, \delta u_1 \sim \text{ev}_1$. Take $a = v_0(u_0)^{-1}\delta^{-1}$ and $b = v_1(u_1)^{-1}\delta^{-1}$. We obtain that fa and gb are homotopic. \square

Proposition 4.15. *The path groupoid $G \ltimes X^I$ is homotopy equivalent to the groupoid $G \ltimes X$. The evaluation $e_1 : G \ltimes X^I \rightarrow G \ltimes X$ is a homotopy equivalence.*

Proof. Consider the map $H : G \ltimes X^I \rightarrow G \ltimes (X^I)^I$ such that $H(\alpha) = \lambda$ with $\lambda : I \rightarrow X^I, \lambda(t) = \alpha(r + t - rt)$. We have the following commutative diagram:

$$\begin{array}{ccccc} G \ltimes X^I & \xleftarrow{\text{ev}_0} & G \ltimes (X^I)^I & \xrightarrow{\text{ev}_1} & G \ltimes X^I \\ & \searrow \text{id} & \uparrow H & \nearrow i \circ e_1 & \\ & & G \ltimes X^I & & \end{array}$$

showing that $i \circ e_1$ is homotopic to the identity map. \square

5. FIBRATIONS

We recall the definition of fibration for topological spaces given as a dualization of the notion of cofibration.

Definition 5.1. [8, 14] A map $p : E \rightarrow B$ is a fibration if for all spaces U with $\text{ev}_0 \circ K = p \circ k$ in the diagram

$$\begin{array}{ccccc} U & & & & \\ & \searrow \tilde{K} & & \searrow K & \\ & E^I & \xrightarrow{p_*} & B^I & \\ & \downarrow \text{ev}_0 & & \downarrow \text{ev}_0 & \\ & E & \xrightarrow{p} & B & \end{array}$$

there exists \tilde{K} that makes the diagram commute.

We want to introduce a notion of fibration for generalized maps. First, let's note that a strict equivariant map $\varphi \ltimes f : G \ltimes X \rightarrow H \ltimes Y$ induces a map $\varphi_* \ltimes f_* : G \ltimes X^I \rightarrow H \ltimes Y^I$ by $f_*(\alpha) = f \circ \alpha$ for all $\alpha \in X^I$ and $\varphi_* = \varphi$ and we proved that,

Proposition 5.2. *If $\epsilon : G \ltimes X \rightarrow H \ltimes Y$ is an essential equivalence, then $\epsilon_* : G \ltimes X^I \rightarrow H \ltimes Y^I$ is an essential equivalence as well.*

Then we have that every generalized map $G \ltimes X \xleftarrow{\epsilon} G' \ltimes X' \xrightarrow{f} H \ltimes Y$ induces a generalized map $G \ltimes X^I \xleftarrow{\epsilon_*} G' \ltimes X'^I \xrightarrow{f_*} H \ltimes Y^I$.

Definition 5.3. A generalized map $G \ltimes X \xleftarrow{\epsilon} G' \ltimes X' \xrightarrow{f} H \ltimes Y$ is a *fibration* if for all translation groupoids $L \ltimes U$ with $\text{ev}_0 \circ (\Omega, K) \Rightarrow (\omega, k) \circ (\epsilon, f)$ in the diagram

$$\begin{array}{ccccccc} L \ltimes U & & & & & & \\ & \searrow \tilde{\Omega} & & \searrow K & & & \\ & \tilde{\mathcal{L}} & & \mathcal{L} & & & \\ & \downarrow \tilde{K} & & \downarrow K & & & \\ & G \ltimes X^I & \xleftarrow{\epsilon_*} & G' \ltimes X'^I & \xrightarrow{f_*} & H \ltimes Y^I & \\ & \downarrow \text{ev}_0 & & \downarrow \text{ev}_0 & & \downarrow \text{ev}_0 & \\ & G \ltimes X & \xleftarrow{\epsilon} & G' \ltimes X' & \xrightarrow{f} & H \ltimes Y & \end{array}$$

there exists $(\tilde{\Omega}, \tilde{K})$ that makes the diagram commute up to 2-isomorphism.

Proof. If the generalized map (ϵ, f) is a fibration, we have that there exists (τ', \tilde{H}') that makes the diagram commute up to 2-isomorphism:

$$\begin{array}{c}
L \ltimes U \xleftarrow{\sigma} \mathcal{L} \\
\swarrow \tau' \quad \searrow \tau \quad \dashrightarrow P \quad \nwarrow H \\
\tilde{\mathcal{L}} \xrightarrow{\tilde{H}'} G \ltimes X^I \xleftarrow{\epsilon_*} G' \ltimes X'^I \xrightarrow{f_*} H \ltimes Y^I \\
\downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \\
\ell \xrightarrow{\epsilon H_0} G \ltimes X \xleftarrow{\epsilon} G' \ltimes X' \xrightarrow{f} H \ltimes Y \\
\curvearrowright H_0
\end{array}$$

$$\begin{array}{ccc} P & \xrightarrow{\tilde{H}''} & G' \ltimes X'^I \\ \downarrow \epsilon'_* & & \downarrow \epsilon_* \\ \tilde{\mathcal{L}} & \xrightarrow{\tilde{H}'} & G \ltimes X^I \end{array}$$

Conversely, if f is a fibration then we have this commutative diagram

$$\begin{array}{c}
L \ltimes U \xleftarrow{\sigma} \mathcal{L} \\
\swarrow \tau \quad \searrow H \\
P \xrightarrow{\tilde{H}'} G' \ltimes X^I \xrightarrow{f_*} H \ltimes Y^I \\
\downarrow \tilde{H} \quad \downarrow \epsilon_* \quad \downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \\
\ell \xrightarrow{H_0} G \ltimes X^I \xleftarrow{\epsilon_*} G' \ltimes X^I \xrightarrow{f_*} H \ltimes Y^I \\
\downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \\
\ell' \xrightarrow{H_0} G \ltimes X \xleftarrow{\epsilon} G' \ltimes X' \xrightarrow{f} H \ltimes Y \\
\downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \quad \downarrow \text{ev}_0 \\
\ell' \xrightarrow{H'_0} G' \ltimes X' \xrightarrow{f} H \ltimes Y
\end{array}$$

Then, the test to decide if a generalized map is a fibration amounts to check the definition of fibration with a strict map. This definition specializes to the following

Definition 5.5. A strict map $G \ltimes X \rightarrow H \ltimes Y$ is a fibration if for all translation groupoids $L \ltimes U$ with $\text{ev}_0 \circ (\Omega, K) \Rightarrow (\omega, k) \circ f$ in the diagram

$$\begin{array}{ccccc}
 L \ltimes U & & & & \\
 \uparrow \omega & \swarrow \tilde{\Omega} & \tilde{\mathcal{L}} & \xrightarrow{\tilde{K}} & \mathcal{L} \\
 & & & & \downarrow K \\
 & & G \ltimes X^I & \xrightarrow{f_*} & H \ltimes Y^I \\
 & & \downarrow \text{ev}_0 & & \downarrow \text{ev}_0 \\
 & & G \ltimes X & \xrightarrow{f} & H \ltimes Y \\
 & \searrow k & & &
 \end{array}$$

there exists $(\tilde{\Omega}, \tilde{K})$ that makes the diagram commute up to 2-isomorphism.

In other words, f is a fibration if for all commutative diagrams

$$\begin{array}{ccc}
 & \tilde{\mathcal{L}} & \\
 \Omega \swarrow & \uparrow \eta & \searrow \text{ev}_0 \circ K \\
 L \ltimes U & \sim & \sim G \ltimes X \\
 \omega \swarrow & \downarrow \nu & \searrow f \circ k \\
 & \ell &
 \end{array}$$

there exists $(\tilde{\Omega}, \tilde{K},)$ such that the following diagrams commute:

$$\begin{array}{ccc}
 & \tilde{K} \ltimes \tilde{Y} & \\
 \tilde{\Omega} \swarrow & \uparrow \eta' & \searrow \text{ev}_0 \circ \tilde{K} \\
 K \ltimes U & \sim & \sim H \ltimes Y^I \\
 \omega \swarrow & \downarrow \nu' & \searrow k \\
 & \ell &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \tilde{K} \ltimes \tilde{Y} & \\
 \tilde{\Omega} \swarrow & \uparrow \eta'' & \searrow f_* \circ \tilde{K} \\
 K \ltimes U & \sim & \sim G \ltimes X \\
 \omega \swarrow & \downarrow \nu'' & \searrow K \\
 & \tilde{\mathcal{L}} &
 \end{array}$$

Explicitly, we have the following

Remark 5.6. A strict map $G \ltimes X \rightarrow H \ltimes Y$ is a fibration if for all $\eta, \nu, \omega, \Omega, K, k$ with $\Omega\eta \sim \omega\nu$ and $\text{ev}_0 K\eta \sim fkv$ there exist $\eta', \nu', \eta'', \nu'', \tilde{\Omega}, \tilde{K}$ such that

$$\begin{aligned}
 \tilde{\Omega}\eta' &\sim \omega\eta' \\
 \tilde{\Omega}\eta'' &\sim \Omega\eta'' \\
 \text{ev}_0 \tilde{K}\eta' &\sim fkv' \\
 f_* \tilde{K}\eta'' &\sim K\nu''
 \end{aligned}$$

where $\eta, \nu, \omega, \Omega, \eta', \nu', \eta'', \nu'', \tilde{\Omega}$ are essential equivalences.

Since a 2-isomorphism between strict maps induces a 2-isomorphism between their path groupoids, we have that being a fibration is a property invariant under 2-isomorphism.

Proposition 5.7. *Consider 2-isomorphic maps $f : G \ltimes X \rightarrow H \ltimes Y$ and $g : G \ltimes X \rightarrow H \ltimes Y$, $f \Rightarrow g$. Then f is a fibration if and only if g is a fibration.*

5.1. The path-loop fibration. Given a point $x \in X$, our various path groupoids are related as follows:

$$\begin{array}{ccccccc} \Omega_x & \longrightarrow & P_x & \longrightarrow & G \ltimes X^I & \xrightarrow{\text{ev}_1} & G \ltimes X \\ & & \downarrow & & \downarrow \text{ev}_0 & & \\ & & \mathbf{1} & \longrightarrow & G \ltimes X & & \end{array}$$

where $\mathbf{1} = e \ltimes x$.

Proposition 5.8 (The path-loop fibration). *The composition $p_1 : P_x \rightarrow G \ltimes X$ given by*

$$P_x \longrightarrow G \ltimes X^I \xrightarrow{\text{ev}_1} G \ltimes X$$

is a fibration.

The groupoid of paths between two points can be found again as the groupoid pullback of the path-loop fibration:

$$\begin{array}{ccc} \Omega_{x,y} & \longrightarrow & P_x \\ \downarrow & & \downarrow p_1 \\ \mathbf{1} & \longrightarrow & G \ltimes X \end{array}$$

where $\mathbf{1} = e \ltimes y$. Indeed, the elements in the object space of this pullback are given by triples $((\alpha, g, w), a, y) \in P_x \times G \times \{y\}$ where a is an arrow between $y \in X$ and w . This is precisely the object space of the groupoid $\Omega_{x,y}$. Indeed, $(\Omega_{x,y})_0 = \{(\beta, k) | \beta(0) = x \text{ and } \beta(1) = ky\}$ and the action is trivial, therefore we have that the groupoid pullback is precisely the based path groupoid $\Omega_{x,y}$ defined in section 3.6.1.

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